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A TRANSFORMATION OF THE HODOGRAPH EQUATION AND THE DETERMINATION OF CERTAIN FLUID MOTIONS

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A transformation is given of the hodograph equation of two-dimensional gas dynamics, from the usual variables q, θ to q and a new variable ϕ. The transformation, which suits any gas for which $p\rho^{-\gamma}=$const. with $\gamma>1$, is so chosen that certain solutions, which in terms of q, θ are multiple-valued, become single-valued functions of q, ϕ. Such a solution is represented, over the whole domain which is of interest, by a single series in q, ϕ, which is rapidly convergent; whereas in terms of q, θ different series would be required for different branches of the function, and these would be but slowly convergent.

By this method we can construct (i) the nozzle flow for which the axial velocity is a prescribed analytic function of position, in particular trans-sonic nozzle flows; and (ii) various cases of flow past aerofoil-shaped cylinders placed in a uniform stream. Taking $\gamma=1{\cdot}4$, complete numerical results are given for one case of trans-sonic nozzle flow, and from these other such flows can be obtained by superposition, and a family of flows of type (ii) is investigated, in which the trailing edge of the aerofoil is cusped; the aerofoil shape has been calculated for two representative values of the free-stream Mach number. A limiting flow of this family is found to consist of a set of Prandtl-Meyer flows, analytically distinct but joining continuously where they abut.

These flows are related to a particular solution of the hodograph equation which is of fundamental analytic importance; it stands in the same relation to the set of 'Chaplygin solutions' as the generating function for Legendre polynomials does to the harmonic functions $r^nP_n(\cos\theta)$.

1. Introduction

In this paper I give a certain transformation of the hodograph equation of gas dynamics, and illustrate its utility for finding flows that are of physical interest. The theory is constructed for the case where the pressure-density relation has the form $p\rho^{-\gamma}=$ const., with $\gamma>1$. The numerical work has been done for the case $\gamma=1{\cdot}4$.

For steady, irrotational, isentropic motion in two dimensions, the hodograph method is to take the velocity components u, v instead of the position co-ordinates x, y as independent variables, the non-linear Eulerian equations of motion in x, y being then equivalent to a *linear* differential equation in u, v; it is from this linearity that the method derives its power. Now, for a physically significant solution, $u(x,y)$, $v(x,y)$ must be single-valued functions, but it may well happen—and for the most important types of flow it does happen—that the inverse functions $x(u,v)$, $y(u,v)$ are multiple-valued; and on the hodograph method we have in the first instance to determine these inverse functions. It is with this problem of determining multiple-valued solutions of the hodograph equation that we shall deal.

The idea of the method is analogous to the familiar one, that a multi-valued relation $w=f(z)$ may sometimes be 'uniformized' by giving both w and z as *single-valued* functions of an auxiliary parameter. For the hodograph case there are two independent variables, conveniently taken as q, θ, the polar components of velocity. Introducing a parameter ϕ we put

$$\theta-\theta_0 = \phi - 2\alpha \arctan \frac{q\sin\phi}{1-q\cos\phi}, \tag{T}$$

where α is a constant simply related to the polytropic exponent γ; and we find that for certain hodograph solutions† x, y are *single-valued* functions of q, ϕ, to be determined by solving a linear differential equation. Elimination of ϕ would give x, y as multi-valued functions of q, θ, but this is unnecessary and, indeed, undesirable; one works throughout with the variables q, ϕ.

Amongst the physically important flows which can be thus determined are (i) the nozzle flow for which the velocity is arbitrarily prescribed on the axis of symmetry—in particular trans-sonic flows, (ii) the 'aerofoil flow' past cylinders of various shapes—in particluar shapes with cusped trailing edge—placed in an infinite stream whose speed at infinity is subsonic. Both analytical and numerical details (correct to at least 0·1%) have been carried through for one example of each of these types (see figures 6, 7, 8), and for the nozzle flow tables are given whence, by superposing other known solutions, an indefinite number of other trans-sonic nozzle flows may be constructed. As an interesting by-product we find (§ 6, figure 9), as a limiting case of one of our solutions, a set of Prandtl-Meyer flows which are analytically distinct, but join continuously where they abut.

In § 4 we investigate (i) a particular hodograph solution which appears to be of fundamental analytic importance and which, accordingly, we call the 'principal solution', and (ii) a related family of solutions. They are given, in terms of q, ϕ, by rapidly convergent series, and the coefficients for three of them are tabulated (tables 1 to 3). Two of them are the physically important solutions referred to in the preceding paragraph. The principal solution stands in the same relation to the 'Chaplygin set' of solutions (in which the variables q, θ are separated) as the generating function for Legendre polynomials does to the harmonic functions $r^n P_n(\cos\theta)$; it is not, like this generating function, elementary, but it has a notably wide domain of regularity in the variables q, ϕ.

A brief comparison may be added with other methods for determining hodograph solutions (Legendre potentials or stream functions) which, as functions of q, θ, are multiple-valued. There are, I think, essentially two of these which are analytically rigorous, and applicable when $p\rho^{-\gamma} = \text{const.}$ (i) The first, due to Bergman (1945, 1948), is to transform the hodograph equation to the form

$$\frac{\partial^2 w}{\partial\lambda^2}+\frac{\partial^2 w}{\partial\theta^2}+wF(\lambda) = 0, \tag{B}$$

where λ is a suitably chosen function of q. (In fact, $\lambda = \mathrm{i}\omega$, where ω is defined in (2·3) below.) For a first approximation the term $wF(\lambda)$ is omitted, so that w can be an arbitrary harmonic function of λ, θ, not necessarily single-valued. An exact solution is then obtained as a series in which this first approximation is the first term. Now $F(\lambda)$ has a singularity at the value of λ corresponding to the sonic value q_s of q; on this account the series cannot converge in a domain including q_s, and for q near q_s it is at best slowly convergent. (ii) The other method (Lighthill 1947; Cherry 1947; Goldstein, Lighthill & Craggs 1948; Tamada 1950) is to

† For the specification of these solutions see the last paragraph of the introduction.

represent the different branches of a multiple-valued solution by series of 'Chaplygin solutions' of the form

$$\Sigma A_\nu f_\nu(q)\, e^{i\nu\theta}. \tag{C}$$

For example, a solution having a simple branch point requires four series of the form (C), analogous to the four expansions in powers of z (two ascending and two descending) for the branches of $(1-z)^{\frac{1}{2}}$; and just as these expansions are only slowly convergent for $|z|$ near 1, so the series (C) are only slowly convergent when q is near its branch value.

Thus both these methods involve the use of slowly convergent series: the slow convergence is not necessarily an analytical impediment, but it is a serious practical one as regards numerical evaluation; and the difficulty is aggravated when the branch value of q is near the sonic value q_s.

The method, (iii) say, of the present paper, is specially adapted to deal with solutions which, as functions of q, θ, have *simple branch points along two characteristics of opposite systems which meet at a point where* $q = q_s$; such solutions become, near these characteristics, single-valued in q, ϕ, and, as the formulae have no singularities at q_s, they are free from the features which in methods (i) and (ii) are associated with slow convergence. But the transformation (T) does not remove any other singularities which may be present, and it not adapted to branch points of higher order. Thus method (iii), so far as it is here developed, is more restricted than (i) or (ii), but in its limited field it seems definitely superior; and this field is one of great physical importance.

Fundamentals

Let $q\cos\theta$, $q\sin\theta$ be rectangular velocity-components at the point x, y, and let Ω be any function (Legendre potential) satisfying the hodograph equation†

$$q^2(1-q^2)\,\Omega_{qq} + (1-q^2/q_s^2)\,(q\Omega_q + \Omega_{\theta\theta}) = 0, \tag{1·1}$$

where q_s, the sonic or critical speed, is related to the polytropic index γ by

$$q_s^2 = \frac{\gamma-1}{\gamma+1}. \tag{1·2}$$

Then, writing

$$X = \Omega_q, \quad Y = q^{-1}\Omega_\theta, \tag{1·3}$$

the solution of the equations

$$x/a = X\cos\theta - Y\sin\theta, \quad y/a = X\sin\theta + Y\cos\theta \tag{1·4}$$

for q, θ in terms of x, y gives a possible flow-field; and all flow-fields other than Prandtl-Meyer expansions or contractions are so obtainable. (The unit of speed is supposed so chosen that $q = 1$ gives the limiting (cavitation) speed for the flow considered; and a is an arbitrary length-constant.) The stream function ψ corresponding to any Ω is found from the consistent equations

$$\psi_q = (1-q^2)^\beta\,(Y - X_\theta), \quad \psi_\theta = -q(1-q^2)^\beta\,(X + Y_\theta). \tag{1·5}$$

The Jacobian of the transformation from q, θ to x, y is given by

$$\frac{1}{a^2}\frac{\partial(x,y)}{\partial(q,\theta)} = X_q(X+Y_\theta) + Y_q(Y+X_\theta) = \frac{q(1-q^2)\,X_q^2}{q^2/q_s^2-1} - qY_q^2 = \frac{q^2/q_s^2-1}{q(1-q^2)}\,(X+Y_\theta)^2 - \frac{(Y-X_\theta)^2}{q}. \tag{1·6}$$

† The similar equation for the stream function ψ is also called the hodograph equation.

Notation

The positive constant q_s is defined in (1·2). Related constants are $\beta = (\gamma-1)^{-1}$; α, the positive root of the equation

$$2\alpha(1+\alpha) = \beta = \frac{1}{\gamma-1} = \frac{1}{2}\left(\frac{1}{q_s^2}-1\right), \tag{1·7}$$

and
$$\lambda = (2\alpha)^{-1}. \tag{1·8}$$

Regarding these we need assume only that $\gamma > 1$, so that β, α, λ are positive.

The symbol ϵ is used, occasionally, as an arbitrarily small positive constant, and A for a positive constant whose precise value is unimportant.

Regarding a single-valued function $w = f(z)$ and its multi-valued inverse $z = f^{-1}(w)$ we use the standard terminology, that a zero z_0 of $f'(z)$ is a *critical point* of $f(z)$, and the corresponding point $w_0 = f(z_0)$ a *branch point* of $f^{-1}(w)$. This entails calling corresponding points by different names, but the alternative of calling them by the same name would seem to be more confusing.

2. The transformation

We are to replace the variable θ in the hodograph equation (1·1) by a related variable ϕ so that certain multiple-valued solutions $\Omega(q, \theta)$ become single-valued functions of q, ϕ. To achieve this the transformation must be properly related to the differential equation. Suppose that a transformation $\theta = f(q, \phi)$, where f is regular and single-valued, has in the $q\phi$-plane a critical locus $\mathscr{L}(\phi)$ on which $\partial f/\partial\phi = 0$. To this corresponds in the $q\theta$-plane a branch locus $\mathscr{L}(\theta)$, and a function $F(q, \phi)$ which is single-valued near $\mathscr{L}(\phi)$ transforms into a function of q, θ which, in general, is branched at every point of $\mathscr{L}(\theta)$. Now it is well known that the only possible branch lines of solutions of a linear differential equation, apart from singular lines of its coefficients, are characteristics of the equation. Hence *if a multi-valued solution $\Omega(q, \theta)$ is to transform into a single-valued function $\Omega(q, f(q, \phi))$, it is necessary that the branch locus $\mathscr{L}(\theta)$ should be characteristic for the hodograph equation*; and the sufficiency of this condition will in due course be verified.

2·1. The transformation to be employed is†

$$\theta + i\kappa = \phi - 2\alpha \arctan\frac{q\sin\phi}{1-q\cos\phi}, \tag{2·1}$$

where the positive constant α has been defined in (1·7) and κ is any constant. From the purely analytical viewpoint the case $\kappa \neq 0$ is converted into the case $\kappa = 0$ by replacing θ by $\theta - i\kappa$, and since this replacement does not affect the hodograph equation (1·1), the two cases are essentially equivalent. But for physical significance we require θ, q to be real, with $0 \leqslant q < 1$, and then, as we shall see, there are two distinct cases, (i) $\kappa = 0$, (ii) κ real and not zero.‡

† In the applications to be made, the inverse tangent will retain its principal value, so θ is effectively a single-valued function of q, ϕ. The transformation is equivalent to one previously given by the author (see Cherry 1947, §6).

‡ The case where $\kappa = \kappa' + i\kappa''$ with $\kappa'' \neq 0$ is reduced to the case $\kappa = \kappa'$ by a real translation on θ, and to this corresponds by (1·4) a rotation of the xy-axes. Hence without loss we can often take $\kappa'' = 0$. [*Added in proof:* However, if a solution of (2·4) is converted by (2·1) into $\Omega(q, \theta, \kappa', \kappa'')$, different values of κ'' give linearly independent functions of q, θ. Hence, for example, we can deduce from table 4 (p. 618) an infinity of independent solutions.]

The critical locus where $\partial\theta/\partial\phi = 0$ is

$$D(q,\phi) \equiv 1-2(1+\alpha)\, q\cos\phi + (1+2\alpha)\, q^2 = 0. \tag{2·2}$$

Solving for ϕ and substituting in (2·1) we obtain, as the branch locus in the $q\theta$-plane,

$$\theta + \mathrm{i}\kappa = \pm\left[\frac{1}{q_s}\arctan\sqrt{\frac{q^2-q_s^2}{1-q^2}} - \arctan\sqrt{\frac{q^2/q_s^2-1}{1-q^2}}\right] = \pm\omega(q), \tag{2·3}$$

where q_s is the 'sonic speed' defined in (1·2); and it is easily verified that this is characteristic for the hodograph equation (1·1). Hence the transformation is suited to the equation. In terms of the variables q, ϕ it becomes

$$L[\Omega] \equiv \{1-2(1+\alpha)\, q\cos\phi + (1+2\alpha)\, q^2\}\left\{q^2\Omega_{qq} + q\Omega_q + \Omega_{\phi\phi} - \frac{2\beta q^3}{1-q^2}\Omega_q\right\}$$
$$+4\alpha q^2 \sin\phi\left(\Omega_{q\phi} - \frac{\beta q\Omega_\phi}{1-q^2}\right) + \frac{4\alpha q(\cos\phi - 2q + q^2\cos\phi)}{1-q^2}\,\Omega_{\phi\phi} = 0, \tag{2·4}$$

where Ω_q means $\partial\Omega(q,\phi)/\partial q$ and not, as in (1·1), $\partial\Omega(q,\theta)/\partial q$. The suitability of the transformation is verified by the fact that, on the critical locus (2·2), the coefficients in (2·4) are regular and those of the second derivatives are not all zero.

In general, a single-valued solution $\Omega(q,\phi)$ of (2·4) is converted by (2·1) into a multiple-valued function of q, θ with branch points on the locus (2·3). It is easily proved that $\partial^2\theta/\partial\phi^2$ vanishes only when $\sin\phi = 0$ or $q = 0, \pm 1$, so the said branch points are in general simple. If $\kappa = 0$ they are real in the supersonic range $q_s \leqslant q < 1$; but if $\kappa \neq 0$ there is only one real branch point, at which $\theta = 0$ and q has the subsonic value satisfying

$$\operatorname{arc\,tanh}\sqrt{\left(\frac{1-q^2/q_s^2}{1-q^2}\right)} - \frac{1}{q_s}\operatorname{arc\,tanh}\sqrt{\left(\frac{q_s^2-q^2}{1-q^2}\right)} = |\kappa|. \tag{2·5}$$

To find the position co-ordinates from a solution $\Omega(q,\phi)$ of (2·4) we use (1·4) along with the transforms of (1·3):

$$X = \Omega_q + \frac{2\alpha\sin\phi\,\Omega_\phi}{D}, \quad Y = \frac{(1-2q\cos\phi + q^2)\,\Omega_\phi}{qD}, \tag{2·6}$$

where D is given by (2·2). The equations (1·5) for ψ can similarly be put in terms of q, ϕ.

The transformation (2·1) can be put in the form

$$\frac{q\,\mathrm{e}^{\kappa-\mathrm{i}\theta}}{q\,\mathrm{e}^{-\mathrm{i}\phi}} = \left(\frac{1-q\,\mathrm{e}^{-\mathrm{i}\phi}}{1-q\,\mathrm{e}^{\mathrm{i}\phi}}\right)^{\alpha}, \tag{2·7}$$

which suggests using $\xi = q\,\mathrm{e}^{\mathrm{i}\phi}$, $\eta = q\,\mathrm{e}^{-\mathrm{i}\phi}$ as alternative variables. With these, (2·4) becomes

$$L_1[\Omega] \equiv \{1-(1+2\alpha)\,\xi\eta\}\,\Omega_{\xi\eta} - \alpha\left(\frac{1-\xi}{1-\eta}\right)(\xi\Omega_{\xi\xi} + \Omega_\xi) - \alpha\left(\frac{1-\eta}{1-\xi}\right)(\eta\Omega_{\eta\eta} + \Omega_\eta)$$
$$-\alpha(1+\alpha)\left\{\frac{1-(1+2\alpha)\,\eta}{1-\eta}\,\xi\Omega_\xi + \frac{1-(1+2\alpha)\,\xi}{1-\xi}\,\eta\Omega_\eta\right\} = 0;$$

or if we use ξ and $\tau = \xi\eta = q^2$ as variables,

$$L_1[\Omega] \equiv \{1-(1+2\alpha)\,\tau\}\,(\tau\Omega_{\tau\tau} + \Omega_\tau + \xi\Omega_{\tau\xi}) - \frac{\alpha(\xi-\tau)}{1-\xi}(\tau\Omega_{\tau\tau} + \Omega_\tau)$$
$$-\frac{\alpha(1-\xi)}{\xi-\tau}(\tau^2\Omega_{\tau\tau} + \tau\Omega_\tau + 2\tau\xi\Omega_{\tau\xi} + \xi^2\Omega_{\xi\xi} + \xi\Omega_\xi)$$
$$-\alpha(1+\alpha)\left\{\frac{1-(1+2\alpha)\,\xi}{1-\xi}\,\tau\Omega_\tau + \frac{\xi-(1+2\alpha)\,\tau}{\xi-\tau}(\tau\Omega_\tau + \xi\Omega_\xi)\right\} = 0. \tag{2·8}$$

2.2. We proceed now to discuss the transformation (2·1) in the physically significant cases, where θ, q are real and $0 \leqslant q < 1$.

Case (i), $\kappa = 0$. It is necessary that θ be real, and (for $0 \leqslant q < 1$) we shall secure this by taking ϕ real; it may also be secured when ϕ is unreal, but this subcase will be noticed under case (ii). Consider the variation of θ with ϕ, for q fixed. The points where $\partial\theta/\partial\phi = 0$, given by (2·2), are real when $q_s \leqslant q < 1$ but unreal for $0 \leqslant q < q_s$. Hence it is easily proved that, taking q, ϕ and q, θ as polar co-ordinates, the circle $q < 1$ in the $q\phi$-plane is in continuous one-one correspondence with a 'Riemann' surface over the $q\theta$-plane (figure 1) which is smooth for $q < q_s$ but is folded like a filter paper for $q_s \leqslant q < 1$; as ϕ increases from $-\pi$ to π, θ increases from $-\pi$ to the value $\omega(q)$ given by (2·3), then decreases to $-\omega(q)$, and then increases to π. Thus for $q_s < q < 1$ and $-\omega(q) < \theta < \omega(q)$, ϕ is a three-valued function of θ, q, but elsewhere in $q < 1$ it is single-valued.

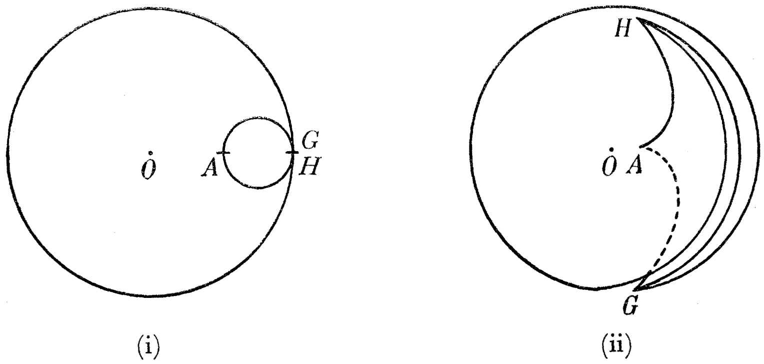


FIGURE 1. (i) $q\phi$-plane, (ii) $q\theta$-surface, for $\kappa = 0$.

Case (ii), $\kappa \neq 0$. For definiteness take κ to be real positive. Since q, θ are to be real, (2·1) cannot be satisfied by a real ϕ. Putting then

$$\phi = \phi' + \mathrm{i}\zeta, \tag{2·9}$$

the separation of (2·1) into real and imaginary parts gives

$$\cos\phi' = \frac{\sinh 2\lambda(\zeta-\kappa) + q^2 \sinh 2(\zeta+\lambda\zeta-\lambda\kappa)}{2q \sinh(\zeta+2\lambda\zeta-2\lambda\kappa)}, \tag{2·10}$$

$$\theta = \phi' - \alpha \arctan \frac{2q\cosh\zeta\sin\phi' - q^2\sin 2\phi'}{1-2q\cosh\zeta\cos\phi' + q^2\cos 2\phi'}, \tag{2·11}$$

where $\lambda = (2\alpha)^{-1}$. For the chosen constant κ, (2·10) gives the connexion between q and the real and imaginary parts of ϕ. It is conveniently represented as a surface on which $q\cos\phi'$, $q\sin\phi'$, ζ are rectangular co-ordinates. Any section $\zeta = $ const. of this surface is a circle having the ends of a diameter at

$$q = \frac{\sinh\lambda(\zeta-\kappa)}{\sinh(\zeta+\lambda\zeta-\lambda\kappa)}, \quad \phi' = 0 \tag{2·12}$$

and

$$q = \frac{\cosh\lambda(\zeta-\kappa)}{\cosh(\zeta+\lambda\zeta-\lambda\kappa)}, \quad \phi' = 0. \tag{2·13}$$

Hence we may prove that the part of the surface for which $q<1$ consists of two detached parts (figure 2); a 'gramophone horn' extending from the circle $\zeta = 2\lambda\kappa(1+2\lambda)^{-1}$, $q = 1$ to a 'vertex' C at $\zeta = +\infty$, $q = 0$, and a 'spindle' extending between vertices E at $\zeta = 0$, $q = 1$, $\phi' = 0$ and F at $\zeta = -\infty$, $q = 0$. The axis $q = 0$ cuts the horn at B, where $\zeta = \kappa$, and the tangent plane at the point (2·12) is parallel to the axis at the point A where ζ satisfies

$$(1+2\lambda)\sinh\zeta = \sinh(\zeta+2\lambda\zeta-2\lambda\kappa), \tag{2·14}$$

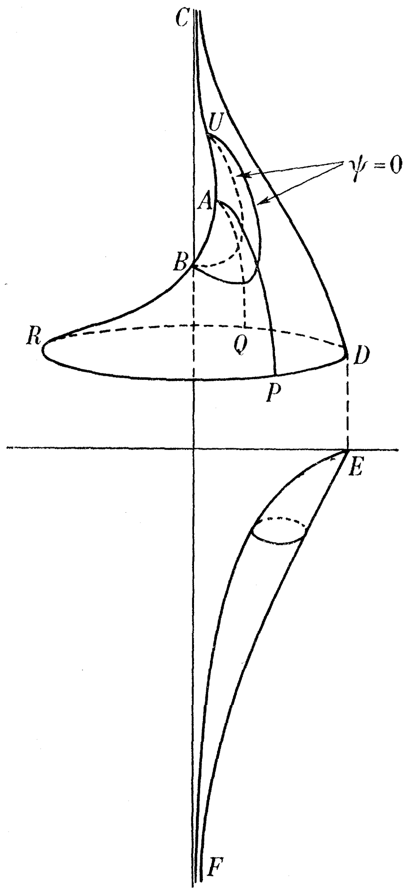


FIGURE 2. Hodograph $(q\phi)$ surface, for $\kappa > 0$.

an equation having just one root. By (2·9), (2·10) we can express D, equation (2·2), in the form

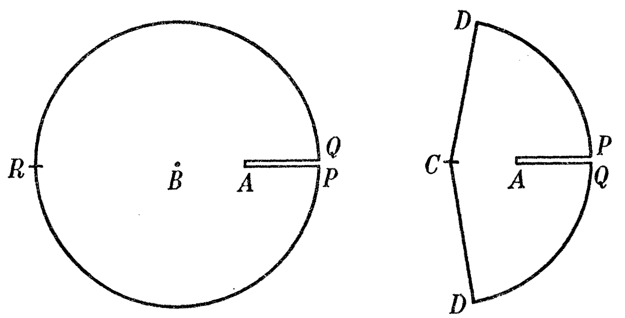
$$D = \alpha(1-q^2)\left\{\frac{(2\lambda+1)\sinh\zeta}{\sinh(\zeta+2\lambda\zeta-2\lambda\kappa)}-1\right\}+2\mathrm{i}(1+\alpha)\,q\sinh\zeta\sin\phi'$$
$$+\frac{2(1+\alpha)\sinh\zeta}{\sinh(\zeta+2\lambda\zeta-2\lambda\kappa)}\{\sinh^2\lambda(\zeta-\kappa)-q^2\sinh^2(\zeta+\lambda\zeta-\lambda\kappa)\}, \tag{2·15}$$

whence it follows that A is on the critical locus $D = 0$, and is the only point of the surface on this locus; and it is easily verified that $q_A < q_s$.

The variation of θ on this surface is given by (2·11). Here the inverse tangent increases by 2π as we circle around any section $\zeta = \text{const.}$, so to make θ single-valued on the surface we cut it down the outer generators CD, EF (the loci of the point (2·13)) and take the inverse tangent to be zero on the inner generators (2·12). Then, taking $q\cos\theta$, $q\sin\theta$ as rectangular co-ordinates, it follows that the horn is in one-one correspondence with a Riemann surface bounded by an arc $q=1$ and rays $\theta=\pm\alpha\pi$ (figure 3);† there is a simple branch point at A, like that on the Riemann surface of $(q\,\mathrm{e}^{\mathrm{i}\theta}-q_A)^{\frac{1}{2}}$, and the principal sheet is regular at B, $q=0$.

The spindle part of the $q\phi$-surface, cut along the outer generator EF, is in one-one correspondence with a sector $q<1$, $-\alpha\pi<\theta<\alpha\pi$, with $q=0$ corresponding to $\zeta=-\infty$.

By taking the limit, as $\kappa\to 0$, of the correspondence just discussed we get a fuller view of case (i). The part of the horn below A shuts down on to the part of the plane $\zeta=0$ between the critical locus (2·2) and the circle $q=1$, while the part above A remains as a horn extending from the critical locus to $\zeta=+\infty$. The spindle degenerates into the conjugate horn together with the interior of the critical locus. These horns, cut along their outer generators, are in one-one correspondence with congruent $q\theta$ regions shown in figure 4, where the rays CH, CG, FH, FG are $\theta=\pm\alpha\pi$ and the curves AH, AG are $\theta=\pm\omega(q)$ (the same curves as in figure 1), meeting the rays where $q=1$.



(i) (ii)

Figure 3. Two sheets of hodograph ($q\theta$) surface, for $\kappa>0$.

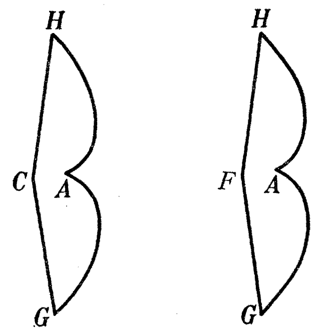


Figure 4. Subsidiary sheets of $q\theta$-surface, for $\kappa=0$.

2·3. *Inversion of the transformation.* It will be sufficient to restrict q to be small. Then for the inverse $\phi=\phi(q,\theta)$ of (2·1) there are three branches, belonging to neighbourhoods of B, C, F in figure 2. (i) For the neighbourhood of B we find the principal branch

$$\phi=\theta+\mathrm{i}\kappa+\sum_{1}^{\infty}\frac{2\Gamma(n\alpha+n)}{n^2\,\Gamma(n)\,\Gamma(n\alpha)}\,q^nF(n\alpha+n,\ -n\alpha;\ n+1;\ q^2)\sin n(\theta+\mathrm{i}\kappa) \tag{2·16}$$

where F denotes the hypergeometirc function. This may be established by seeking the Fourier series for $\phi-\theta-\mathrm{i}\kappa$, as in the solution of Kepler's equation. (ii) Near the point C, where $q\sim 0$ and $\zeta\sim+\infty$, (2·7) gives the first approximation $q\,\mathrm{e}^{-\mathrm{i}\phi}\sim 1$, and higher approximations to the solution are easily deduced, e.g.

$$q\,\mathrm{e}^{-\mathrm{i}\phi}=1-(1-q^2)\,(q\,\mathrm{e}^{\kappa-\mathrm{i}\theta})^{1/\alpha}+O(q\,\mathrm{e}^{\kappa-\mathrm{i}\theta})^{2/\alpha}. \tag{2·17}$$

(iii) For the third branch (neighbourhood of F) there are similar approximations.

† The polar angles ϕ', θ are not single-valued on the respective surfaces, and for the rays to be described as $\theta=\pm\alpha\pi$ we should cut the surfaces along BR. In (2·12) we cover all cases by the convention that, on BR, $\phi'=0$ and q is negative.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

3. General solution for trans-sonic nozzle flow

It is desired to find the flow in a two-dimensional nozzle, with an axis of symmetry, for which on the axis the speed q is a prescribed function of the position co-ordinate x. Let this function $q(x)$ be analytic, and strictly increasing over the whole or part of the range $0 \leqslant q < 1$ as x increases over some finite or infinite range. Then x is an analytic function of q, say $x = f'(q)$, regular and increasing in the whole or part of $0 \leqslant q < 1$; for simplicity of statement let it be the whole range. Taking the velocity-direction θ to vanish on the axis, the potential Ω of the desired flow is to be an even function of θ, and by (1·3) and (1·4) we are to have $\Omega_q = f'(q)$ when $\theta = 0$.

Hence Ω is to be that solution of the hodograph equation (1·1) for which, on the axis $\theta = 0$, the Cauchy data are $\Omega = f(q)$, $\Omega_\theta = 0$. Since the coefficient of $\Omega_{\theta\theta}$ is non-zero for $0 \leqslant q < q_s$ but vanishes for $q = q_s$, the desired solution certainly exists and is regular for $0 \leqslant q \leqslant q_s - \epsilon$ and θ sufficiently small, but the question of its continuation as q increases past q_s presents a difficulty. This difficulty is overcome if we use the variables q, ϕ (with $\kappa = 0$ in (2·1)) instead of q, θ, and find Ω from the transformed equation (2·4). Supposing at first that $0 \leqslant q < q_s$, Ω is to be an even function of ϕ, with $\Omega = f(q)$ for $\phi = 0$. However, for $\phi = 0$ the coefficient of $\Omega_{\phi\phi}$ becomes $(1-q)^2(1+q+2\alpha q)/(1+q)$, which is regular and non-zero in the whole range $0 \leqslant q < 1$. Hence the Cauchy data $\Omega = f(q)$, $\Omega_\phi = 0$ extend over the whole range, and there is a solution Ω regular for $0 \leqslant q \leqslant 1-\epsilon$ and ϕ sufficiently small.

The position co-ordinates x, y are now to be determined via (2·6), and we may prove as follows that the apparent singularity of these formulae on the critical locus $D = 0$ is illusory. For when (q, ϕ) is confined to this locus we have

$$\frac{d\Omega_\phi}{dq} = \Omega_{\phi q} + \Omega_{\phi\phi}\frac{d\phi}{dq}, \quad \frac{d\phi}{dq} = \frac{(1+\alpha)\cos\phi - (1+2\alpha)\,q}{(1+\alpha)\,q\sin\phi},$$

and thence we find that (2·4) reduces on the locus to

$$\frac{d\Omega_\phi}{dq} = \frac{\beta q}{1-q^2}\,\Omega_\phi.$$

Hence, on the locus, $\Omega_\phi = C(1-q^2)^{-\frac{1}{2}\beta}$, where C is constant; at the point of the locus where $q = q_s$, $\phi = 0$ and $\Omega_\phi = 0$ since Ω is even; so $C = 0$ and Ω_ϕ vanishes on the locus. And finally, since the functions are analytic, Ω_ϕ/D must be regular on the locus.

Regarding the correspondence between the xy- and $q\phi$-planes, we have on the axis $\phi = 0$, by (1·6) and (2·1),

$$\frac{1}{a^2}\frac{\partial(x,y)}{\partial(q,\phi)} = \frac{q(1-q^2)\,X_q^2}{q^2/q_s^2-1}\,\frac{1-2(1+\alpha)\,q+(1+2\alpha)\,q^2}{(1-q)^2} = -\frac{q(1+q)\,X_q^2}{1+q/q_s},$$

and here $X_q = f''(q)$, which by hypothesis is regular and non-zero. Hence, near the axis, the correspondence between the two planes is one-one; and by § 2 the xy-plane is in one-one correspondence with the $q\theta$ surface of figure 1, so that, for example, any supersonic velocity (q, θ) for which $|\theta| < \omega(q)$ occurs at three distinct points (x, y). In particular, the locus $\theta = 0$ consists of the axis $y = 0$ together with a curve cutting the axis at the point where $q = q_s$; and hence follows the well-known convergent-divergent pattern of the streamlines.

To calculate Ω we may substitute in (2·4) a development $\Omega = f(q) + \phi^2 f_1(q) + \phi^4 f_2(q) + \ldots$ and equate coefficients of powers of ϕ. Hence, in succession, $f_1, f_2, \ldots$ are determined in terms of the datum function $f'(q)$ and its derivatives. Alternatively, a development in powers of $\sin\phi$ or $\tan\frac{1}{2}\phi$ could be used. For such developments to be of practical use it is necessary that the convergence be reasonably fast when ϕ is 'not too small'. This matter will not be pursued here, in view of the suggestion that (2·4) invites solution by a Fourier series in ϕ rather than by a power series, and that a Fourier solution may be expected to converge uniformly for all real ϕ. In § 4 this suggestion will be followed up, and § 5 will give a special case of nozzle-flow to which the investigation leads.

4. The principal solution

The hodograph equation (1·1) has elementary solutions

$$e^{\pm i\nu\theta}\, q^\nu F(a_\nu, b_\nu; \nu+1; q^2), \tag{4·1}$$

where the parameter ν can have any value other than a negative integer, and

$$a_\nu = \tfrac{1}{2}(\nu+\beta) + \tfrac{1}{2}\surd\{(1+2\beta)\,\nu^2+\beta^2\}, \quad b_\nu = \tfrac{1}{2}(\nu+\beta) - \tfrac{1}{2}\surd\{(1+2\beta)\,\nu^2+\beta^2\}; \tag{4·2}$$

as usual, F denotes the hypergeometric function. We shall choose a certain linear combination of these solutions, (4·19) below, express it by means of (2·1) in terms of q, ϕ, and prove that the resulting function is regular, apart from poles on the critical locus (2·2), in a domain of the complex variables q, ϕ which includes the segment $0 \leqslant q < 1$ and all real ϕ. This is an indirect method of attacking the differential equation (2·4), whose justification is that it penetrates deeper than does the direct attack.

Some preliminary clearing of ground (§§ 4·1 to 4·3) is necessary. For shortness we write

$$F(a_\nu, b_\nu; \nu+1; q^2) = F_\nu(q^2). \tag{4·3}$$

4·1. *The poles of $F_\nu(q^2)$, and the function h_ν.* As a function of ν (for fixed q), $F_\nu(q^2)$ is meromorphic, with poles at $\nu = -1, -2, \ldots$. Its residue at $\nu = -n$ is (Cherry 1947)

$$-h_n q^{2n} F_n(q^2), \tag{4·4}$$

where
$$h_\nu = \frac{\Gamma(a_\nu)\,\Gamma(1+\nu-b_\nu)}{\Gamma(a_\nu-\nu)\,\Gamma(1-b_\nu)\,\Gamma(\nu)\,\Gamma(1+\nu)}. \tag{4·5}$$

In (4·4) there is involved only the value of h_ν when ν is a positive integer n, and (4·5) then reduces to a form symmetrical in a_n, b_n—as is necessary since $F(a_n, b_n; n+1; q^2)$ is symmetrical in these arguments. For general ν, however, h_ν is unsymmetrical, and the convention is that in the definitions (4·2) of a_ν, b_ν the radical is real positive when ν is real positive, so that then a_ν is real positive and $a_\nu > b_\nu$. On account of this radical, h_ν has simple branch points where

$$\nu = \pm i\beta(1+2\beta)^{-\frac{1}{2}} = \pm ic.$$

We shall take the ν-plane to be cut *between* these points; then in the cut plane a_ν, b_ν, h_ν are single-valued, and we shall require to know the poles and zeros of h_ν. These occur only where one of the arguments ν, a_ν, $a_\nu-\nu$, $1+\nu-b_\nu$, $1-b_\nu$ is zero or a negative integer. Writing $\nu = \xi + i\eta$, we find from (4·2) that b_ν is real on the axis $\eta = 0$ and also on the ellipse

$$\xi^2(4+2/\beta) + 2\eta^2/\beta = 1,$$

whose foci are at the branch points $\nu = \pm ic$; where the ellipse cuts the real axis, b_ν is stationary. As ν moves along the real axis from $+\infty$ to $(4+2/\beta)^{-\frac{1}{2}}$, thence round either half of the ellipse to $-(4+2/\beta)^{-\frac{1}{2}}$, and thence to $-\infty$, b_ν increases steadily from $-\infty$ to $+\infty$, its values on the ellipse ranging between the positive values

$$\tfrac{1}{2}\beta\left(1-\sqrt{\frac{2\beta}{1+2\beta}}\right), \quad \tfrac{1}{2}\beta\left(1+\sqrt{\frac{2\beta}{1+2\beta}}\right).$$

Since (4·2) gives $a_\nu - \nu = \beta - b_\nu$, $a_\nu - \nu$ runs on the ellipse between the same two values. Hence the following facts:

(i) *For ν on the positive real axis, h_ν has neither poles nor zeros;*

(ii) *On the ellipse, h_ν has no poles, and has zeros at the conjugate pairs of points where $b_\nu = 1, 2, \ldots$, which exist if $\beta \geqslant (2+4\sqrt{2})/7$; one or more pairs of these zeros is in the right half-plane if $\beta > 2$; for $\beta = 2\cdot5$ they are $\nu = 0\cdot1 \pm 1\cdot091\mathrm{i}$ (where $b_\nu = 1$), and $\nu = -0\cdot3 \pm 0\cdot843\mathrm{i}$ (where $b_\nu = 2$).*

(iii) *On the negative real axis h_ν has an infinity of both poles and zeros.*

The behaviour of h_ν when $|\nu|$ is large may be found by noting that (4·2) give

$$a_\nu = \nu(1+\alpha) + \tfrac{1}{2}\beta + O(\nu^{-1}), \quad b_\nu = -\nu\alpha + \tfrac{1}{2}\beta + O(\nu^{-1}) \tag{4·6}$$

and thence, with Stirling's theorem, approximating to the gamma functions in (4·5); we obtain, for $|\arg \nu| \leqslant \pi - \epsilon$,

$$2\pi h_\nu = \delta^{-2\nu}\{1+O(\nu^{-1})\}, \tag{4·7}$$

where

$$\delta = \alpha^\alpha(1+\alpha)^{-1-\alpha}. \tag{4·8}$$

Also (4·2) give, in the cut plane,

$$a_{-\nu} = b_\nu - \nu, \quad b_{-\nu} = a_\nu - \nu,$$

and thence follows

$$h_\nu h_{-\nu} = -\frac{\sin(a_\nu - \nu)\pi \sin b_\nu \pi \sin^2 \nu\pi}{\pi^2 \sin a_\nu \pi \sin(b_\nu - \nu)\pi}. \tag{4·9}$$

4·2. *A Fourier series related to the transformation* (2·1). From (2·7) we obtain

$$\mathrm{e}^{\mathrm{i}\nu\theta - \nu\kappa - \mathrm{i}\nu\phi} = (1-q\,\mathrm{e}^{\mathrm{i}\phi})^{\nu\alpha}(1-q\,\mathrm{e}^{-\mathrm{i}\phi})^{-\nu\alpha}, \tag{4·10}$$

ν being any constant. Expanding by the binomial theorem on the right and collecting like powers of $\mathrm{e}^{\mathrm{i}\phi}$ gives

$$\mathrm{e}^{\mathrm{i}\nu\theta-\nu\kappa} = \sum_{r=0}^{\infty} \mathrm{e}^{\mathrm{i}(\nu+r)\phi} \frac{\Gamma(r-\nu\alpha)}{\Gamma(-\nu\alpha)\,\Gamma(r+1)} q^r F(\nu\alpha, r-\nu\alpha; r+1; q^2)$$
$$+ \sum_{r=1}^{\infty} \mathrm{e}^{\mathrm{i}(\nu-r)\phi} \frac{\Gamma(r+\nu\alpha)}{\Gamma(\nu\alpha)\,\Gamma(r+1)} q^r F(-\nu\alpha, r+\nu\alpha; r+1; q^2);$$

it may be noted that $\Gamma(r\pm\nu\alpha)/\Gamma(\pm\nu\alpha)$ is a polynomial in ν, and hence is non-singular when $r\pm\nu\alpha$ is zero or a negative integer.

Now when r is a positive integer, and ν is arbitrary,

$$\lim_{s\to r} \frac{\Gamma(-\nu\alpha-s)}{\Gamma(-\nu\alpha)\,\Gamma(1-s)} q^{-s} F(\nu\alpha, -s-\nu\alpha; 1-s; q^2) = \frac{\Gamma(\nu\alpha+r)}{\Gamma(\nu\alpha)\,\Gamma(r+1)} q^r F(-\nu\alpha, r+\nu\alpha; r+1; q^2), \tag{4·11}$$

and on the left we can without ambiguity replace s by r and suppress the limit sign; by symmetry, the resulting formula is then valid when r is *any* integer (the case $r = 0$ being evident).

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Substituting in the preceding equation, it becomes, for the case where ν is an integer n (not necessarily positive),

$$\mathrm{e}^{\mathrm{i}n\theta-n\kappa} = \sum_{r=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}(n-r)\phi} \frac{\Gamma(r+n\alpha)}{\Gamma(n\alpha)\,\Gamma(r+1)}\, q^r F(-n\alpha, r+n\alpha; r+1; q^2)$$

$$= \sum_{r=-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}r\phi} \frac{\Gamma(r+n+n\alpha)}{\Gamma(n\alpha)\,\Gamma(r+n+1)}\, q^{r+n} F(-n\alpha, r+n+n\alpha; r+n+1; q^2). \tag{4·12}$$

Supposing ϕ to be real, the expansion on the right of (4·10) gives a double series, the moduli of whose terms have a sum less than

$$(1-|q|)^{-2\alpha|\nu|}, \tag{4·13}$$

and putting $\nu = n$, this is a bound for the sum of the moduli of the terms on the right of (4·12). Thus (4·12) is valid for ϕ real and $|q|<1$.

4·3. *Asymptotic formulae for $F_\nu(q^2)$, etc.* (i) The behaviour of $F_\nu(q^2)$, defined by (4·3), when ν is large has been investigated by several authors (e.g. Cherry 1947); the result which will be subsequently needed is: *when $0 \leqslant q \leqslant q_s - \epsilon$, and ν is large but excluded from small neighbourhoods of the negative integers,*

$$F_\nu(q^2) = \frac{(1-q^2)^{\frac{1}{4}-\frac{1}{2}\beta}}{(1-q^2/q_s^2)^{\frac{1}{4}}}\, q^{-\nu}\delta^{\nu}\,\mathrm{e}^{-\nu u}\{1+O(\nu^{-1})\}, \tag{4·14}$$

where δ is defined in (4·8) and

$$u = \operatorname{arc\,tanh}\sqrt{\left(\frac{1-q^2/q_s^2}{1-q^2}\right)} - \frac{1}{q_s}\operatorname{arc\,tanh}\sqrt{\left(\frac{q_s^2-q^2}{1-q^2}\right)} > 0.$$

(ii) For the hypergeometric function which occurs in (4·12), investigations similar to those just mentioned give: *Under the same conditions on q, ν,*

$$F(-\nu\alpha, r+\nu+\nu\alpha; r+\nu+1; q^2) = \left(\frac{1-q^2}{1-q^2/q_s^2}\right)^{\frac{1}{4}} Q^r q^{-\nu}\delta^{\nu}\,\mathrm{e}^{-\nu u}\{1+O(\nu^{-1})\}, \tag{4·15}$$

the remainder-term being uniform when r is a fixed integer but not when $r \sim \infty$; here

$$Q = \frac{\sqrt{(1-q^2)}-\sqrt{(1-q^2/q_s^2)}}{\sqrt{(1-q^2)}-\sqrt{(q_s^2-q^2)}}\,\frac{q_s}{q^2(1+\alpha)}. \tag{4·16}$$

(iii) From the formula (2·16) it is seen that the function $F(\nu+\nu\alpha, -\nu\alpha; \nu+1; q^2)$ is intimately bound up with the transformation (2·1), and from (4·6) we see that it is in some sense an approximation to $F_\nu(q^2)$. From (4·11), its residue at the pole $\nu = -n$ is

$$-h_n^*\, q^{2n} F(n\alpha+n, -n\alpha; n+1; q^2),$$

where

$$h_\nu^* = \frac{\Gamma(\nu\alpha+\nu)\,\Gamma(\nu\alpha+\nu+1)}{\Gamma(\nu\alpha)\,\Gamma(\nu\alpha+1)\,\Gamma(\nu)\,\Gamma(\nu+1)}, \tag{4·17}$$

and the analogue of (4·7), for $\nu \sim \infty$ with $|\arg\nu| \leqslant \pi-\epsilon$, is

$$2\pi h_\nu^* = \delta^{-2\nu}\{1+O(\nu^{-1})\}. \tag{4·18}$$

4·4. *The principal solution of the hodograph equation.* We define this as

$$\Omega_0 = \sum_{n=1}^{\infty} \frac{\Gamma(n\alpha+n)}{\Gamma(n\alpha+1)\,\Gamma(n)} \left(\mathrm{e}^{-\mathrm{i}n\theta+n\kappa} + \frac{h_n\,\mathrm{e}^{\mathrm{i}n\theta-n\kappa}}{h_n^*}\right) q^n F_n(q^2), \tag{4·19}$$

a linear combination of the elementary solutions (4·1), and shall express it in terms of q, ϕ. The *a priori* reason for supposing this solution to be of importance is not relevant to what follows, but is sketched in appendix 2. We shall prove (§§ 4·4, 4·5) the following

THEOREM. *The function* Ω_0 *under the transformation* (2·1) *is identical in the sense of analytic continuation with*

$$\Omega_0 = \frac{(1-q^2)^{1-\frac{1}{2}\beta}}{(1+\alpha)\{1-2(1+\alpha)\,q\cos\phi+(1+2\alpha)\,q^2\}} - \frac{1}{1+\alpha} + G(q^2, q\,e^{i\phi}), \tag{4·20}$$

where the function G, given explicitly by (4·33), *is regular provided* (i) $\mathscr{R}q^2 < 1$ *and* $\mathscr{R}(q\,e^{i\phi}) < 1$, *or* (ii) $0 \leqslant q^2 < 1$ *and* $q\,e^{i\phi}$ *is not on the ray* $1 \leqslant q\,e^{i\phi} < +\infty$.

(i) By Stirling's theorem, $\Gamma(n\alpha+n)/\Gamma(n\alpha+1)\,\Gamma(n) = O(\delta^{-n}n^{-\frac{1}{2}})$, and by (4·7) and (4·18), $h_n/h_n^* = O(1)$. Also by (4·14), for $0 \leqslant q < q_s$, $q^nF_n(q^2) = O(\delta^n\,e^{-nu})$, so when θ is real the series in (4·19) is dominated by $\Sigma An^{-\frac{1}{2}}\,e^{-nu}$, and converges absolutely since $u > 0$. Substitute for $e^{\pm in(\theta+i\kappa)}$ from (4·12). In view of (4·13), the resulting double series is dominated by one whose sum is $\sum_1^\infty An^{-\frac{1}{2}}\,e^{-nu}(1-q)^{-2\alpha n}$, so it converges absolutely when $e^{-u} < (1-q)^{2\alpha}$, which is true for q sufficiently small (since $q \to 0$ gives $u \to +\infty$). Hence when θ, ϕ are real and q is positive and sufficiently small we can rearrange the double series in the form

$$\Omega_0 = \sum_{r=-\infty}^{\infty} e^{-ir\phi}\,C_r(q), \tag{4·21}$$

where

$$q^{-r}C_r(q) = \sum_{n=1}^{\infty} \frac{\Gamma(n\alpha+n)\,\Gamma(r-n\alpha-n)\,F_n(q^2)\,F(n\alpha, r-n\alpha-n; r-n+1; q^2)}{\Gamma(n\alpha+1)\,\Gamma(-n\alpha)\,\Gamma(n)\,\Gamma(r+1-n)}$$
$$+ \sum_{n=1}^{\infty} \frac{h_n\Gamma(n\alpha+n)\,\Gamma(r+n\alpha+n)\,q^{2n}F_n(q^2)\,F(-n\alpha, r+n\alpha+n; r+n+1; q^2)}{h_n^*\,\Gamma(n\alpha+1)\,\Gamma(n\alpha)\,\Gamma(n)\,\Gamma(r+1+n)}. \tag{4·22}$$

(ii) When r is a positive integer we can find a closed formula for $C_r(q)$. Consider, as a function of ν,

$$f_{r,\nu}(q^2) = -\Gamma(1-\nu)\frac{\Gamma(r-\nu-\nu\alpha)}{\Gamma(1-\nu-\nu\alpha)}\,F_\nu(q^2)\,\frac{F(\nu\alpha, r-\nu-\nu\alpha; r-\nu+1; q^2)}{\Gamma(r-\nu+1)}. \tag{4·23}$$

The first factor has simple poles at $\nu = 1, 2, 3, \ldots$, the second is regular, the third has simple poles at $\nu = -1, -2, -3, \ldots$ and the fourth is regular. Since

$$\frac{\Gamma(1-\nu)}{\Gamma(1-\nu-\nu\alpha)} = \frac{\Gamma(\nu+\nu\alpha)\sin\nu(1+\alpha)\pi}{\Gamma(\nu)\sin\nu\pi}, \qquad \frac{1}{\Gamma(n\alpha+1)\,\Gamma(-n\alpha)} = -\frac{\sin n\alpha\pi}{\pi},$$

the residue at $\nu = n > 0$ is equal to the term shown in the first line on the right of (4·22); and from (4·4) and (4·17) the residue at $\nu = -n$ is the term shown on the second line. At $\nu = 0$, $f_{r,\nu}(q^2)$ is regular. For $\nu \sim \infty$, with r fixed,

$$\frac{\Gamma(1-\nu)\,\Gamma(r-\nu-\nu\alpha)}{\Gamma(1+r-\nu)\,\Gamma(1-\nu-\nu\alpha)} = \frac{(r-1-\nu-\nu\alpha)\ldots(1-\nu-\nu\alpha)}{(r-\nu)\ldots(1-\nu)} = -\frac{(1+\alpha)^{r-1}}{\nu}\{1+O(\nu^{-1})\},$$

so from (4·14) and (4·15)

$$f_{r,\nu}(q^2) = \frac{(1-q^2)^{\frac{1}{2}-\frac{1}{2}\beta}}{(1-q^2/q_s^2)^{\frac{1}{2}}}\,\frac{Q^r(1+\alpha)^{r-1}}{\nu}\{1+O(\nu^{-1})\},$$

provided ν is bounded from the integers. Hence, integrating $f_{r,\nu}(q^2)$ round an infinite circle,

$$C_r(q) = \frac{(1-q^2)^{\frac{1}{2}(1-\beta)}}{(1-q^2/q_s^2)^{\frac{1}{2}}(1+\alpha)}\{(1+\alpha)\,qQ\}^r \quad (r>0). \tag{4·24}$$

(iii) When r is zero or a negative integer $-s$, $f_{-s,\nu}(q^2)$ has (from its second factor) additional poles at $\nu = 0,\ -1/(1+\alpha),\ \ldots,\ -s/(1+\alpha)$. Taking account of the new residues, and writing $(1+\alpha)^{-1} = \mu$, we get

$$C_{-s}(q) = \frac{\mu(1-q^2)^{\frac{1}{2}(1-\beta)}}{(1-q^2/q_s^2)^{\frac{1}{2}}}\left(\frac{qQ}{\mu}\right)^{-s} + \sum_{k=0}^{s}\frac{(-1)^{s-k+1}\,\mu\Gamma(1+k\mu)\,q^{-s}F_{-k\mu}(q^2)\,F(k\mu-k,k-s;\,k\mu-s+1;\,q^2)}{\Gamma(k\mu-s+1)\,(s-k)!\,k!}. \tag{4·25}$$

When $s>0$ this formula masks the regularity of $C_{-s}(q)$ at $q=0$, which is directly evident from (4·19) and (4·21), and a better formula is obtained as follows:

(iv) By appeal to (4·11), with $r = s\pm n$ and $\nu = \pm n$, (4·22) is converted into

$$q^{-s}C_{-s}(q) = \sum_{n=1}^{\infty}\frac{\Gamma(n\alpha+n)\,\Gamma(s+n\alpha+n)\,q^{2n}F_n(q^2)\,F(-n\alpha,s+n\alpha+n;\,s+n+1;\,q^2)}{\Gamma(n\alpha+1)\,\Gamma(n\alpha)\,\Gamma(n)\,\Gamma(s+n+1)} + \sum_{n=1}^{\infty}\frac{h_n\Gamma(n\alpha+n)\,\Gamma(s-n\alpha-n)\,F_n(q^2)\,F(n\alpha,s-n\alpha-n;\,s-n+1;\,q^2)}{h_n^*\,\Gamma(n\alpha+1)\,\Gamma(-n\alpha)\,\Gamma(n)\,\Gamma(s-n+1)},$$

a formula, in which s is a positive integer, closely resembling (4·22). The second term on the right is the sum of the residues of $h_\nu f_{s,\nu}(q^2)/h_\nu^*$ at $\nu = 1, 2, \ldots$, and the first term is the sum of the residues of $h_{-\nu}^* f_{s,\nu}(q^2)/h_{-\nu}$ at $\nu = -1, -2, \ldots$. By (4·17) h_ν^* has neither poles nor zeros in the right half-plane, so from § 4·1 h_ν/h_ν^* is regular in this half-plane; while $h_{-\nu}^*/h_{-\nu}$ is regular in the left half-plane, except for simple poles at the zeros of $h_{-\nu}$ which lie there when $\beta>2$. Also h_ν/h_ν^*, $h_{-\nu}^*/h_{-\nu}$ are $1+O(\nu^{-1})$ at infinity in the respective half-planes.

By Cauchy's theorem, therefore, we obtain the analogue of (4·24):

$$C_{-s}(q) = \frac{(1-q^2)^{\frac{1}{2}(1-\beta)}}{(1-q^2/q_s^2)^{\frac{1}{2}}(1+\alpha)}\{(1+\alpha)\,qQ\}^s + q^s g_s(q^2) \quad (s>0), \tag{4·26}$$

where
$$g_s(q^2) = \frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty(-)}\frac{h_{-\nu}^* f_{s,\nu}(q^2)\,\mathrm{d}\nu}{h_{-\nu}} - \frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty(+)}\frac{h_\nu f_{s,\nu}(q^2)\,\mathrm{d}\nu}{h_\nu^*}; \tag{4·27}$$

here $f_{s,\nu}$ is defined in (4·23), the notation $\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty(-)}$ means that the far parts of the path lie on the imaginary axis but that the central part passes to the left of the branch points $\pm\mathrm{i}c$ and of the unreal zeros of $h_{-\nu}$, and for $\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty(+)}$ the path similarly passes to the right of the branch points; and both integrals are principal values, $\lim_{\mu\to\infty}\int_{-\mathrm{i}\mu}^{\mathrm{i}\mu}$.

(v) When we substitute from (4·24) and (4·26) and (for $s=0$) (4·25) into (4·21) and sum the terms involving Q, we obtain the formula (4·20), with

$$G(q^2, q\,\mathrm{e}^{\mathrm{i}\phi}) = \sum_1^{\infty} q^n g_n(q^2)\,\mathrm{e}^{\mathrm{i}n\phi}. \tag{4·28}$$

This has been established for θ, ϕ real and q sufficiently small positive, conditions which are sufficient for the identity of the two functions, in the sense of analytic continuation. Regarding this continuation, the essential thing is of course to discuss the function $G(q^2, q\,\mathrm{e}^{\mathrm{i}\phi})$.

4·5. *Analytic continuation of* $G(q^2, q\,\mathrm{e}^{\mathrm{i}\phi})$. (i) The desired continuation is achieved by means of a trivial adjustment of the formula (4·27) for $g_n(q^2)$. Choosing two points on the imaginary axis beyond $\nu = \pm \mathrm{i}c$, say $\nu = \pm 2\mathrm{i}c$, (4·27) can be written

$$g_n(q^2) = \frac{1}{2\pi\mathrm{i}}\int_{-2\mathrm{i}c}^{2\mathrm{i}c\,(-)} \frac{h^*_{-\nu} f_{n,\nu}(q^2)\,\mathrm{d}\nu}{h_{-\nu}} - \frac{1}{2\pi\mathrm{i}}\int_{-2\mathrm{i}c}^{2\mathrm{i}c\,(+)} \frac{h_\nu f_{n,\nu}(q^2)\,\mathrm{d}\nu}{h^*_\nu} + \frac{1}{2\pi\mathrm{i}}\left(\int_{2\mathrm{i}c}^{\mathrm{i}\infty} + \int_{-\mathrm{i}\infty}^{-2\mathrm{i}c}\right)\left(\frac{h^*_{-\nu}}{h_{-\nu}} - \frac{h_\nu}{h^*_\nu}\right) f_{n,\nu}(q^2)\,\mathrm{d}\nu. \quad (4·29)$$

Now (4·17) gives
$$h^*_\nu h^*_{-\nu} = -\frac{\sin^2 \nu\alpha\pi \sin^2 \nu\pi}{\pi^2 \sin^2 \nu(1+\alpha)\pi},$$
and thence from (4·9)

$$\frac{h_\nu h_{-\nu}}{h^*_\nu h^*_{-\nu}} = \frac{\sin(a_\nu - \nu)\pi \sin b_\nu\pi \sin^2\nu(1+\alpha)\pi}{\sin a_\nu\pi \sin(b_\nu - \nu)\pi \sin^2 \nu\alpha\pi}$$
$$= 1 - \frac{\sin \nu\pi\{\sin(a_\nu - b_\nu - 2\alpha\nu - \nu)\pi \cos\nu\pi - \sin(a_\nu - b_\nu)\pi + \cos\beta\pi \sin\nu(1+2\alpha)\pi\}}{2\sin a_\nu\pi \sin(b_\nu - \nu)\pi \sin^2\nu\alpha\pi};$$

and by (4·6) this is $1 + O(\sec 2\nu\alpha\pi)$ for $\nu \sim \pm\mathrm{i}\infty$. Since also $h^*_{-\nu}/h_{-\nu} = 1 + O(\nu^{-1})$, we have

$$\frac{h^*_{-\nu}}{h_{-\nu}} - \frac{h_\nu}{h^*_\nu} = O(\sec 2\nu\alpha\pi). \quad (4·30)$$

Thus by combining the integrals as in (4·29) *we replace factors which for* $\nu \sim \pm\mathrm{i}\infty$ *are only* $O(1)$ *by a factor which is exponentially small*. The remaining factors, involving q^2, can thus become large without destroying the convergence.

For shortness, now, let us denote the right-hand member of (4·29) by

$$\frac{1}{2\pi\mathrm{i}}\int \left(\frac{h^*_{-\nu}}{h_{-\nu}} \text{ or } \frac{h_\nu}{h^*_\nu}\right) f_{n,\nu}(q^2)\,\mathrm{d}\nu. \quad (4·29\,\textit{bis})$$

Substitute the value (4·23) of $f_{n,\nu}(q^2)$, and observe that

$$\frac{\Gamma(n-\nu-\nu\alpha)}{\Gamma(n-\nu+1)} F(\nu\alpha, n-\nu-\nu\alpha; n-\nu+1; q^2) = \frac{\Gamma(-\nu\alpha)}{2\pi\mathrm{i}}\int_0^{(1+)} t^{n-\nu(1+\alpha)-1}(t-1)^{\nu\alpha}(1-q^2t)^{-\nu\alpha}\,\mathrm{d}t, \quad (4·31)$$

provided (as is the case for all n, ν here in question) the real part of $n-\nu(1+\alpha)$ is positive; the path of integration is to have $t=1$ inside it and $t=q^{-2}$ outside. Thus

$$G(q^2, q\,\mathrm{e}^{\mathrm{i}\phi}) = \sum_1^\infty q^n g_n(q^2)\,\mathrm{e}^{\mathrm{i}n\phi}$$
$$= -\frac{1}{2\pi\mathrm{i}}\int\left(\frac{h^*_{-\nu}}{h_{-\nu}} \text{ or } \frac{h_\nu}{h^*_\nu}\right)\frac{\Gamma(1-\nu)\,\Gamma(-\nu\alpha)\,F_\nu(q^2)\,\mathrm{d}\nu}{2\pi\mathrm{i}\,\Gamma(1-\nu-\nu\alpha)}\sum_1^\infty q^n \mathrm{e}^{\mathrm{i}n\phi}\int_0^{(1+)} t^{n-\nu(1+\alpha)-1}(t-1)^{\nu\alpha}(1-q^2t)^{-\nu\alpha}\,\mathrm{d}t, \quad (4·32)$$

the summation under the integral sign being justified by absolute convergence when ϕ is real and $0 \leqslant q \leqslant q_s - \epsilon$. Hence, under the conditions for which (4·20) was proved,

$$G(q^2, q\,e^{i\phi}) = -\frac{1}{2\pi i}\int\left(\frac{h^*_{-\nu}}{h_{-\nu}} \text{ or } \frac{h_\nu}{h^*_\nu}\right)\frac{\Gamma(1-\nu)\,\Gamma(-\nu\alpha)\,F_\nu(q^2)\,d\nu}{2\pi i\,\Gamma(1-\nu-\nu\alpha)}\int_0^{(1+)}\frac{t^{-\nu(1+\alpha)}(t-1)^{\nu\alpha}\,(1-q^2t)^{-\nu\alpha}\,dt}{q^{-1}\,e^{-i\phi}-t} \tag{4·33}$$

where the notation means the sum of four integrals as in (4·29).

(ii) To establish the regularity of G for $\mathscr{R}q^2 < 1$ and $\mathscr{R}q\,e^{i\phi} < 1$ it is sufficient to prove that, of these four integrals, those along the imaginary axis to $\pm i\infty$ converge uniformly provided

$$|\,q^{-2}-\tfrac{1}{2}-\epsilon\,| \geqslant \tfrac{1}{2}+2\epsilon, \quad |\,q^{-1}\,e^{-i\phi}-\tfrac{1}{2}-\epsilon\,| \geqslant \tfrac{1}{2}+2\epsilon, \tag{4·34}$$

for any positive ϵ. By substituting for $F_\nu(q^2)$ from a formula resembling (4·31) the integrand becomes

$$\left(\frac{h^*_{-\nu}}{h_{-\nu}} - \frac{h_\nu}{h^*_\nu}\right)\frac{\Gamma(1-\nu)\,\Gamma(-\nu\alpha)\,\Gamma(1+\nu)\,\Gamma(a_\nu-\nu)}{4\pi^2\,\Gamma(1-\nu-\nu\alpha)\,\Gamma(a_\nu)}\,J_1 J_2, \tag{4·35}$$

where

$$J_1 = \int_0^{(1+)}\frac{t^{-\nu(1+\alpha)}(t-1)^{\nu\alpha}\,(1-q^2t)^{-\nu\alpha}\,dt}{q^{-1}\,e^{-i\phi}-t}, \quad J_2 = \int_0^{(1+)} t^{\alpha_\nu-1}(t-1)^{\nu-\alpha_\nu}\,(1-q^2t)^{-b_\nu}\,dt. \tag{4·36}$$

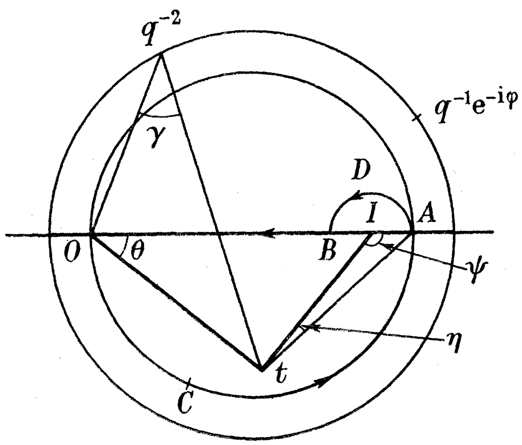

FIGURE 5. Path of integration in t-plane.

We first find a bound for J_1. When $\nu = i\mu$, with μ real, the modulus of its integrand is $|\,q^{-1}\,e^{-i\phi}-t\,|^{-1}\,e^{\mu S}$, where

$$S = (1+\alpha)\arg t - \alpha\arg(t-1) + \alpha\arg(1-q^2t)$$
$$= -(1+\alpha)\,\theta + \alpha\psi + \alpha\gamma,$$

the notation being that of figure 5. Here the points O, B, I, A are $t = 0,\ 1-2\epsilon,\ 1,\ 1+2\epsilon$, so that the circle OCA is $|\,t-\tfrac{1}{2}-\epsilon\,| = \tfrac{1}{2}+\epsilon$; the outer circle is $|\,t-\tfrac{1}{2}-\epsilon\,| = \tfrac{1}{2}+2\epsilon$ and by (4·34) the points q^{-2}, $q^{-1}\,e^{-i\phi}$ are not within it; the angles θ, ψ, γ are positive in the figure, but may vary to negative values as q^{-2}, t move.

For t on the lower semicircle OCA, OtA is a right angle, so $S = -(\theta+\alpha\eta)+\alpha\gamma+\tfrac{1}{2}\alpha\pi$; also $\gamma < \tfrac{1}{2}\pi$ since q^{-2} is outside the circle OCA, and $\theta+\alpha\eta$ has a positive lower bound.† Hence

$$S < \alpha\pi - \kappa_\epsilon \quad (\kappa_\epsilon > 0). \tag{4·37}$$

† Either θ or η is not less than its value when It is perpendicular to OA, so

$$\theta+\alpha\eta > \arctan(2\epsilon)^{\frac{1}{2}}\max(1, \alpha) = \kappa_\epsilon.$$

For t on the upper semicircle ADB, centre I, $S = (1+\alpha)\,\theta'+\alpha\psi+\alpha\gamma$, where $\theta' = -\theta \geqslant 0$, $\psi \leqslant 0$, and $\gamma < \frac{2}{3}\pi$.† Hence $S < \frac{5}{6}\alpha\pi$ provided $(1+\alpha)\,\theta' < \frac{1}{6}\alpha\pi$, and since $\theta' \leqslant \arcsin 2\epsilon$ this condition is secured by choosing ϵ sufficiently small.

For t on the segment BO, $\psi = -\pi$, $\theta = 0$ and $\gamma < \pi$, so $S < 0$. Hence the condition (4·37) is satisfied, for t on the path $OCADBO$, provided q^{-2} satisfies $(4{\cdot}34)_1$; so for μ positive,

$$\exp(\mu S) < \exp \mu(\alpha\pi - \kappa_\epsilon).$$

Also $(4{\cdot}34)_2$ gives, for t on this path, $|\,q^{-1}\,e^{-i\phi}-t\,| \geqslant \epsilon$; so for $\nu = i\mu$ with μ real positive,

$$|\,J_1\,| \leqslant \pi(1+2\epsilon)\,\epsilon^{-1} \exp\{|\,\nu\,|\,(\alpha\pi-\kappa_\epsilon)\}. \tag{4·38}$$

When μ is negative the same bound for J_1 is obtained by integrating along the reflexion of the path $OCADBO$ in the real axis.

A bound for J_2 is easily deduced by comparing the two integrands in (4·36). We have

$$a_\nu = \nu(1+\alpha)+\tfrac{1}{2}\beta+O(\nu^{-1}), \quad b_\nu = -\nu\alpha+\tfrac{1}{2}\beta+O(\nu^{-1}),$$

and hence

$$|\,J_2\,| \leqslant A \exp\{|\,\nu\,|\,(\alpha\pi-\kappa_\epsilon)\} \int_0^{(1+)} |\,t^{\frac{1}{2}\beta-1}(t-1)^{-\frac{1}{2}\beta}\,(1-q^2t)^{-\frac{1}{2}\beta}\,|\,|\,dt\,|,$$

where the integral is finite since $\beta > 0$. Finally, by Stirling's theorem the gamma-function factor in (4·35) is proved to be bounded for $\nu \sim \pm i\infty$. In view of (4·30), therefore, the integrand is $O\{\exp -|\,2\nu\kappa_\epsilon\,|\}$ when ν is pure imaginary, and this establishes the desired uniform convergence under the conditions (4·34), and thence the regularity of $G(q^2, q\,e^{i\phi})$ for $\mathscr{R}q^2 < 1$ and $\mathscr{R}q\,e^{i\phi} < 1$.

It is clear that we can enlarge the domain of one of q^{-2}, $q^{-1}\,e^{-i\phi}$ and retain the uniform convergence provided we suitably contract the domain of the other. In particular, if q^2 is real, with $0 \leqslant q^2 \leqslant 1-\epsilon$, the path for J_1 can be taken to and fro along the segment OI, and if $q^{-1}\,e^{-i\phi}$ is distant at least ϵ from this, $|\,J_1\,| \leqslant 2\epsilon^{-1} \exp\{|\,\nu\,|\,\alpha\pi\}$. The estimate for J_2 is unchanged, and the uniform convergence follows.

The theorem stated in § 4·4 is thus established.

4·6. *Asymptotic behaviour of* $g_n(q^2)$ *for* $n \sim \infty$. For simplicity let $0 \leqslant q^2 \leqslant 1-\epsilon$. In (4·29) let the paths for the first two integrals be moved to the imaginary axis, with infinitesimal indentations round $\nu = \pm ic$. If $\beta > 2$, the path for the first integral, during its displacement, crosses two or more zeros $-\nu_k$ of $h_{-\nu}$, so we get

$$g_n(q^2) = \sum_k \frac{h^*_{\nu_k} f_{n,\,-\nu_k}(q^2)}{h'_{\nu_k}} + \text{integrals} \quad (h'_\nu = dh_\nu/d\nu).$$

The integrals, whose integrands have the form shown in (4·32) with the symbols $\Sigma q^n\,e^{in\phi}$ omitted, may be estimated as in § 4·5; using the result

$$\left|\int_0^{(1+)} t^{n-\nu(1+\alpha)-1}(t-1)^{\nu\alpha}\,(1-q^2t)^{-\nu\alpha}\,dt\right| \leqslant (2/n)\exp\{|\,\nu\,|\,\alpha\pi\},$$

which (by integration along the real axis) is valid for $q^2 < 1$ and ν pure-imaginary, the integrals are proved to be uniformly $O(n^{-1})$.

† This is a crude estimate; at the point q^{-2}, a radius of the circle ADB can subtend at most $\frac{1}{6}\pi$, and OI can subtend at most $\frac{1}{2}\pi$.

In the first term on the right, substitute the value (4·23) of $f_{n,-\nu_k}(q^2)$, and observe that for n large and ν bounded

$$\left.\begin{aligned} F(-\nu\alpha, n+\nu+\nu\alpha;\, n+\nu+1;\, q^2) &= (1-q^2)^{\nu\alpha}\{1+O(n^{-1})\},\\ \frac{\Gamma(n+\nu+\nu\alpha)}{\Gamma(n+\nu+1)} &= n^{\nu\alpha-1}\{1+O(n^{-1})\}. \end{aligned}\right\} \tag{4·39}$$

We obtain

$$g_n(q^2) = -\sum_k \frac{h^*_{\nu_k}\Gamma(1+\nu_k)}{h'_{\nu_k}\Gamma(1+\nu_k+\nu_k\alpha)}(1-q^2)^{\nu_k\alpha}F_{-\nu_k}(q^2)\,n^{\nu_k\alpha-1}+O(n^{-1}), \tag{4·40}$$

which is the desired asymptotic formula.

If $\beta<2$ the first term on the right is absent, and $g_n(q^2)=O(n^{-1})$. If $\beta>2$ the dominant terms are the pair for which $\mathscr{R}\nu_k$ is largest, and since $\mathscr{I}\nu_k\neq 0$, g_n is an oscillatory function of n, whose order lies between n^{-1} and $n^{\frac{1}{2}\alpha-1}$, since (from § 4·1), $\mathscr{R}\nu_k<\frac{1}{2}$.

Essentially because of (4·39), an improved estimate of $g_n(q^2)$ is to be sought by displacing the 'finite parts' of the paths of integration in (4·29) as far as possible to the right. For the first integral the path gets caught on the branch points $\nu=\pm ic$, and it is from these that the principal part of the integral is contributed. Hence the remainder term in (4·40) can be shown to be in fact $O\{n^{-1}(\log n)^{-\frac{3}{2}}\}$.

4·7. *Calculation of* $g_n(q^2)$. The best method is to use the differential equation (2·8). Substituting the expression on the right of (4·20) we obtain

$$L_1[G] = -\beta\tau(2-\beta\tau)(1-\tau)^{-1-\frac{1}{2}\beta}/(1+\alpha) \quad (\tau=q^2),$$

and then putting $G=\sum_1^\infty \xi^n g_n(\tau)$ and equating coefficients of powers of ξ we get the general recurrence relation

$$\begin{aligned}(1+\alpha)(1-\tau)\tau^2g_n''&+\{(n+1+2n\alpha+\alpha)(1-\tau)-2\alpha(1+\alpha)^2\tau\}\tau g_n'+n\alpha\{n-(1+\alpha)(1+2\alpha)\tau\}g_n\\ &=\{1+(1+2\alpha)\tau\}\{(1-\tau)(\tau g''_{n-1}+ng'_{n-1})-\alpha(1+\alpha)(2\tau g'_{n-1}+(n-1)g_{n-1})\}+2\alpha(n-1)^2g_{n-1}\\ &\quad-(1+\alpha)(1-\tau)\tau g''_{n-2}-\{(n-1+\alpha)(1-\tau)-2\alpha(1+\alpha)^2\tau\}g'_{n-2}-\alpha(n-2)(n-3-\alpha)g_{n-2}.\end{aligned} \tag{4·41}$$

This is valid as it stands for $n\geqslant 3$, for $n=2$ it is valid if we put $g_0=0$, and for $n=1$ we replace the right-hand member by

$$-\beta\tau(2-\beta\tau)(1-\tau)^{-1-\frac{1}{2}\beta}/(1+\alpha).$$

Hence we can solve successively for $g_1, g_2, \ldots$. From § 4·5 it is known that the g_n are regular in $|\tau|<1$, and the condition of regularity at $\tau=0$ suffices to determine them uniquely from the differential equations. Thus we find, in the first instance,

$$g_1(\tau)=\frac{1}{(1+\alpha)^2\tau}\{(1-\tau)^{1-\frac{1}{2}\beta}-F_{-\mu}(\tau)\}, \quad \mu=(1+\alpha)^{-1}. \tag{4·42}$$

Writing

$$g_n(\tau)=\sum_{r=0}^\infty a_{n,r}\tau^r \tag{4·43}$$

the relation (4·41) gives

$$\begin{aligned}A_{n,r}a_{n,r}&-B_{n,r-1}a_{n,r-1}\\ &=C_{n-1,r+1}a_{n-1,r+1}+D_{n-1,r}a_{n-1,r}-E_{n-1,r-1}a_{n-1,r-1}-F_{n-2,r+1}a_{n-2,r+1}+G_{n-2,r}a_{n-2,r},\end{aligned} \tag{4·44}$$

where $A_{n,r}$, $B_{n,r}$, ..., $G_{n,r}$ are quadratic polynomials in n, r, and we are to take $a_{n,r} = 0$ for $n<1$ or $r<0$.

If we start from the first p coefficients in the series for g_1, the relations (4·44) determine $p-1$ coefficients for g_2, $p-2$ for g_3, etc., and finally one for g_p. A check on the calculations is provided by the identity (4·45) below. For the case $\beta = 2{\cdot}5$ the coefficients as far as $p = 15$ are given in table 1†; these are sufficient to give $G(q^2, q\,e^{i\phi})$ correct to 4 decimal places when $0 \leqslant q^2 \leqslant 0{\cdot}4$ and ϕ is real, and to three places when $0{\cdot}4 \leqslant q^2 \leqslant 0{\cdot}5$.

4·8. *Properties of* $G(q^2, q\,e^{i\phi})$. (i) It is significant to show *that G is regular for the unreal values of ϕ that came into question in* § 2, *q being real and less than* 1. For this it is, by the theorem of § 4·4, sufficient that $q\,e^{i\phi}$ be not real and on $(1, +\infty)$.

Writing $\phi = \phi' + i\zeta$ we have $q\,e^{i\phi} = q\,e^{-\zeta}\,e^{i\phi'}$, so as regards the 'horn' part of figure 2, where $\zeta > 0$, the result is immediate. On the inner generator of the 'spindle' we have from (2·12)

$$q\,e^{i\phi} = \frac{e^{-\zeta}\sinh\lambda(\kappa-\zeta)}{\sinh(\lambda\kappa-\lambda\zeta-\zeta)} = 1 - \frac{e^{\lambda\zeta-\lambda\kappa}\sinh(-\zeta)}{\sinh(\lambda\kappa-\lambda\zeta-\zeta)},$$

which is between 0 and 1, since $\zeta < 0$. On the outer generator, similarly, $q\,e^{i\phi} > 1$, and elsewhere $q\,e^{i\phi}$ is unreal. Hence the desired result is established provided the spindle is cut along its outer generator, as we have supposed in § 2.

(ii) The value of $G(q^2, q^2)$ may be obtained as follows: From the form of (4·19) the mean value of the right-hand member of (4·20) with respect to θ is 0. Since

$$\frac{\partial\theta}{\partial\phi} = \frac{1-2(1+\alpha)\,q\cos\phi+(1+2\alpha)\,q^2}{1-2q\cos\phi+q^2} = 1 - \frac{\alpha q\,e^{i\phi}}{1-q\,e^{i\phi}} - \frac{\alpha q\,e^{-i\phi}}{1-q\,e^{-i\phi}},$$

$$\int_{-\pi}^{\pi} G(q^2, q\,e^{i\phi})\,d\theta = -\alpha\int_{-\pi}^{\pi}\frac{\sum_{n=1}^{\infty} q^n g_n(q^2)\,e^{in\phi}\,q\,e^{-i\phi}\,d\phi}{1-q\,e^{-i\phi}} = -2\pi\alpha\sum_1^{\infty} q^{2n} g_n(q^2).$$

After similar integration of the other terms, and transposing, we get

$$G(q^2, q^2) = \sum_1^{\infty} q^{2n} g_n(q^2) = \frac{(1-q^2)^{-\frac{1}{2}\beta}-1}{\alpha(1+\alpha)}. \tag{4·45}$$

(iii) The limit of $G(q^2, q\,e^{i\phi})$ at the point C of figure 2, where $q \sim 0$, $q\,e^{i\phi} \sim 0$, is immediately evident, $G \sim 0$. To find its behaviour near F, where $q \sim 0$, $q\,e^{-i\phi} \sim 0$ and $q\,e^{i\phi} \sim 1$, let the path for the inner integral in (4·33) be displaced so that it encloses the point $t = q^{-1}\,e^{-i\phi}$. We must subtract the residue $(q\,e^{i\phi})^{\nu}\,(1-q\,e^{i\phi})^{\nu\alpha}\,(1-q\,e^{-i\phi})^{-\nu\alpha}$, which by (2·6) equals $(q\,e^{i\theta-\kappa})^{\nu}$, and hence

$$G(q^2, q\,e^{i\phi}) = \frac{1}{2\pi i}\int\left(\frac{h^*_{-\nu}}{h_{-\nu}}\text{ or }\frac{h_\nu}{h^*_\nu}\right)\frac{\Gamma(1-\nu)\,\Gamma(-\nu\alpha)\,F_\nu(q^2)\,d\nu}{\Gamma(1-\nu-\nu\alpha)}$$
$$\times\left\{(q\,e^{i\theta-\kappa})^{\nu} + \frac{1}{2\pi i}\int_0^{(1+)}\frac{t^{-\nu(1+\alpha)}\,(t-1)^{\nu\alpha}\,(1-q^2t)^{-\nu\alpha}\,dt}{t-q^{-1}\,e^{-i\phi}}\right\}. \tag{4·46}$$

The contribution from the new inner integral is regular near $q^2 = 0$, $q\,e^{i\phi} = 1$, and its limit is $O(1)$. For the remaining term (interpreted as in (4·29 *bis*), let both paths of integration be moved into the right half-plane, except for indentations around the branch points

† All the tables are at the end of the paper, pp. 615 *et seq.*

$\nu = \pm ic$; on doing this we must subtract residues at the zeros $-\nu_k$ of $h_{-\nu}$. Then let $q \to 0$ through positive real values, while θ remains fixed. The integral becomes $O\{(\log q)^{-\frac{3}{2}}\}$, and the residues become $O(q^{-\nu_k})$. Thus we find

$$G = \sum_k \frac{h^*_{\nu_k} \Gamma(1+\nu_k)\,\Gamma(\nu_k \alpha)}{h'_{\nu_k} \Gamma(1+\nu_k+\nu_k\alpha)} (q\,e^{i\theta-\kappa})^{-\nu_k} + O(1), \tag{4·47}$$

and if $\beta > 2$, so that $\mathscr{R}\nu_k > 0$, the limit of G is ∞ as $q \to 0$.

4·9. *Solutions related to the principal solution.* For any positive integer p, the hodograph solution

$$\Omega_p = \sum_1^\infty \frac{\Gamma(n\alpha+n)}{n^p \Gamma(n\alpha+1)\,\Gamma(n)} \left(e^{-in(\theta+i\kappa)} + \frac{(-1)^p h_n\, e^{in(\theta+i\kappa)}}{h^*_n}\right) q^n F_n(q^2) \tag{4·48}$$

can be expressed, as in the preceding work, in the form

$$\Omega_p = \sum_0^\infty q^n g^{(p)}_{-n}(q^2)\, e^{-in\phi} + \sum_1^\infty q^n g^{(p)}_n(q^2)\, e^{in\phi}, \tag{4·49}$$

where the last term has the same domain of regularity as $G(q^2, q\,e^{i\phi})$ in the theorem of § 4·4, and (for $n \geq 0$) $g^{(p)}_{-n}$ is the residue at $\nu = 0$ of

$$\frac{\Gamma(1-\nu)\,\Gamma(n-\nu-\nu\alpha)\,F_\nu(q^2)\,F(\nu\alpha, n-\nu-\nu\alpha;\, n-\nu+1;\, q^2)}{\nu^p\,\Gamma(1-\nu-\nu\alpha)\,\Gamma(1+n-\nu)}.$$

In particular,

$$g^{(1)}_0(q^2) = -\frac{1}{2(1+\alpha)} \int_0^{q^2} \frac{(1-\tau)^{-\beta}-1}{\tau}\, d\tau, \quad g^{(1)}_{-n}(q^2) = \frac{1}{n} \quad (n \geq 1), \tag{4·50}$$

$$\left.\begin{aligned} g^{(2)}_0(q^2) &= \alpha\Big\{3q^2 + \frac{q^4}{2^2} + \frac{q^6}{3^2} + \dots + \frac{1+\beta}{1!}\left(\frac{\frac{1}{2}\beta}{1(1+\beta)} + \frac{3}{2}\right)\frac{q^4}{2^2} \\ &\qquad + \frac{(1+\beta)\,(2+\beta)}{2!}\left(\frac{\frac{1}{2}\beta}{1(1+\beta)} + \frac{\frac{1}{2}\beta}{2(2+\beta)} + \frac{4}{3}\right)\frac{q^6}{3^2} + \dots\Big\}, \\ g^{(2)}_{-n}(q^2) &= \frac{1}{n^2} - \frac{\alpha}{n}\left(1 + \frac{1}{2} + \dots + \frac{1}{n-1}\right) + \alpha\left(\frac{q^2}{1(1+n)} + \frac{q^4}{2(2+n)} + \dots\right) \\ &\qquad + \frac{1}{2n}\int_0^{q^2} \frac{(1-\tau)^{-\beta}-1}{\tau}\, d\tau \quad (n \geq 1). \end{aligned}\right\} \tag{4·51}$$

A convenient method of calculating the coefficients in (4·49) follows from the fact that

$$\Omega_p = i\frac{\partial \Omega_{p+1}}{\partial \theta} = \frac{i(1-2q\cos\phi+q^2)}{D} \frac{\partial \Omega_{p+1}}{\partial \phi}. \tag{4·52}$$

Suppose the coefficients in Ω_p known. Then $D\Omega_p/(1-2q\cos\phi+q^2)$ can be expanded as a Fourier series, and this must lack the absolute term since it is the ϕ-derivative of the Fourier series $i\Omega_{p+1}$; this gives the condition

$$g^{(p)}_0 = \alpha \sum_1^\infty q^{2n}(g^{(p)}_{-n} + g^{(p)}_n), \tag{4·53}$$

analogous to (4·45). The integration of (4·52) now determines $\Omega_{p+1} - g^{(p+1)}_0$, and finally $g^{(p+1)}_0$ is determined from (4·53), with $p+1$ in place of p. Thus the calculation of $\Omega_1, \Omega_2, \dots$ can be based upon the known Ω_0. The coefficients for Ω_1, Ω_2 are given in tables 2 and 3, for the case $\gamma = 1·4$, $\beta = 2·5$.

Starting from Ω_0, further solutions $\Omega_{-1}, \Omega_{-2}, \ldots$ can be derived by differentiation according to (4·52); Ω_{-p} has poles of order $2p+1$ on the critical locus $D = 0$.

From (2·6) and (1·5), along with (4·52), it is verified that the values of X, Y, ψ associated with the solution Ω_p are

$$\left.\begin{aligned} Y_p &= -\mathrm{i}q^{-1}\Omega_{p-1}, \quad X_p = \frac{\partial \Omega_p}{\partial q} - \frac{2\mathrm{i}\alpha\Omega_{p-1}\sin\phi}{1-2q\cos\phi+q^2}, \\ \psi_p &= (1-q^2)^{\beta}\left\{-\mathrm{i}q\frac{\partial\Omega_{p+1}}{\partial q} - \frac{2\alpha q\Omega_p\sin\phi}{1-2q\cos\phi+q^2} + \mathrm{i}\Omega_{p-1}\right\}. \end{aligned}\right\} \tag{4·54}$$

Amongst the properties of the solutions Ω_p are:

(i) *For* $p \geqslant 2$, $\partial\Omega_p/\partial\phi$ *vanishes on the critical locus.* This follows from (4·52) by transposing the factor D, and then putting $D = 0$. For $p = 1$, $D\Omega_0$ has a finite limit, whence

$$\partial\Omega_1/\partial\phi = -\mathrm{i}(1-q^2)^{-\frac{1}{2}\beta}/\alpha \quad \text{on } D = 0. \tag{4·55}$$

(ii) At the point $q = q_s$, $\phi = 0$, for $p \geqslant 3$, $\partial^2\Omega_p/\partial q^2$ vanishes, while for $p = 2$

$$\frac{\partial^2\Omega_2}{\partial q^2} = \frac{2(1-q_s^2)^{-\frac{1}{2}\beta}}{q_s^2(1-q_s)}. \tag{4·56}$$

This follows from (1·1), which for $\phi = 0$ becomes

$$\frac{\partial^2\Omega_p}{\partial q^2} = \frac{1-q^2/q_s^2}{q^2(1-q^2)}\left(\Omega_{p-2} - q\frac{\partial\Omega_p}{\partial q}\right).$$

5. A case of trans-sonic nozzle flow

In § 3 it has been seen that a potential $\Omega(q, \phi)$ which is even in ϕ specifies a trans-sonic nozzle flow provided that for $\phi = 0$ it is regular and its derivative Ω_q (which gives the position co-ordinate x on the axis) is increasing over some range of q which includes the sonic point q_s. From § 4·9 it is seen that the real part of Ω_2 satisfies these conditions; the crucial point is that by (4·56) its second derivative is positive at $\phi = 0$, $q = q_s$. The associated values of X, Y, ψ are given by (4·54), with $p = 2$; and the position co-ordinates x, y follow from (1·4), where $\theta(q, \phi)$ is given by (2·1) with $\kappa = 0$. From these formulae (taking the real parts) the flow field has been calculated for a fluid (say air) for which $\gamma = 1·4$, and a representative set of streamlines is plotted in figure 6. Taking any symmetrical pair of the streamlines as rigid boundaries, we have here a nozzle in which the axial velocity increases rather more slowly than x, and in which consequently the supersonic part is quite slowly divergent.

The interest of this example is that the approximation, to better than 1 in 1000, is uniform over the whole field, so that we are enabled (i) to determine how far the regular flow-field extends from the axis of the nozzle, and the manner in which the regularity finally disappears, (ii) to check the accuracy of approximate nozzle theories.

Regarding (i): As each streamline is followed back from the throat into the subsonic region, the speed q on it diminishes to a minimum and then again increases to supersonic values; and here, for q a little greater than q_s, the regular flow terminates in a limit-line. On streamlines which are sufficiently remote from the axis the flow is entirely supersonic.

Figure 6 shows also, as of mathematical interest, the analytic continuation of the nozzle flow into new sheets of the flow-plane. At the point in the hodograph plane (approx. $q/q_s = 1{\cdot}43$, $\theta = 116°$) corresponding to A the stream function is a local minimum, and near A the flow-plane has four sheets which abut in pairs at four limit-lines; near A these are, in pairs, almost coincident, and the figure does not separate them. This type of singularity has been noticed by Craggs (1948, p. 378).

Regarding (ii), the accuracy of approximate nozzle theories: The simple 'hydraulic' theory of Reynolds, and a modification of it proposed by Friedrichs, have been checked on the streamlines $\psi = 0{\cdot}1$ and $\psi = 0{\cdot}3$ of figure 6, out to $M = 2{\cdot}24$ ($q/q_s = 1{\cdot}73$). Both theories give the formula

$$S^{-1} = Cq(1-q^2)^{\beta},$$

where C is constant on a streamline; with Reynolds, S is the transverse co-ordinate y to the streamline from the axial point where the speed is q; with Friedrichs, S is the length of a circular arc, orthogonal to the streamline and to the axis, through the streamline-point where the speed is q (the underlying assumption is that such arcs should be approximately the loci of constant speed). *In the supersonic part* of the nozzle, out to $\psi = 0{\cdot}3$, the Reynolds theory is correct to about 0·2 %; it gives y a little too large for M round about 1·3, and a little too small for M round about 1·8. The Friedrichs theory has its largest discrepancy, about 0·4 %, round $M = 1{\cdot}3$, and for M round 1·8 is rather better than Reynolds'. *In the subsonic part*, we clearly cannot expect such good agreement. In the Reynolds theory, the percentage errors in y, in excess, are about as follows

$M=$	0·745	0·565	0·456
$\psi=0{\cdot}1$	0·3	0·8	2·5
$\psi=0{\cdot}3$	3·7	7·6	13

For the Friedrichs theory the errors are considerably larger, as we might expect since the constant-speed loci are grossly discrepant from orthogonality to the streamlines.

From this case of nozzle flow, an indefinite number of others can be found by superposing other hodograph solutions, and for this purpose the values of ψ, X, Y, for the solution $\mathscr{R}\Omega_2$ of figure 6, are given in table 4, for $q^2 = 0(0{\cdot}02)0{\cdot}50$ and ϕ at degree intervals over a sufficient range.† The associated values of θ are readily found from (2·1), and a short table only (table 5) is given to show its general march.

6. A family of 'aerofoil-type' flows

Consider an infinite stream, in steady two-dimensional irrotational isentropic flow past a fixed cylinder; we call this a flow of 'aerofoil-type'. Let the velocity q_∞ at infinity be subsonic and parallel to Ox, and let the flow be symmetrical about Ox. It is well known that the Legendre potential $\Omega(q,\theta)$ of such a flow is even in θ, and has at $q = q_\infty$, $\theta = 0$ a branch point where its first derivatives are infinite like the real part of $(1-q\,e^{i\theta}/q_\infty)^{-\frac{1}{2}}$. In § 2 it has been shown that, in general, single-valued solutions $\Omega(q,\phi)$ of (2·4) are converted by the

† The relevant range of ϕ decreases as q increases. The author has a MS. table for the complete range $0 \leqslant \phi \leqslant 180°$, at degree intervals, and could supply photo copies of it. It was computed by the Mathematics Division of the Commonwealth Scientific and Industrial Research Organization, Australia, using punched-card methods.

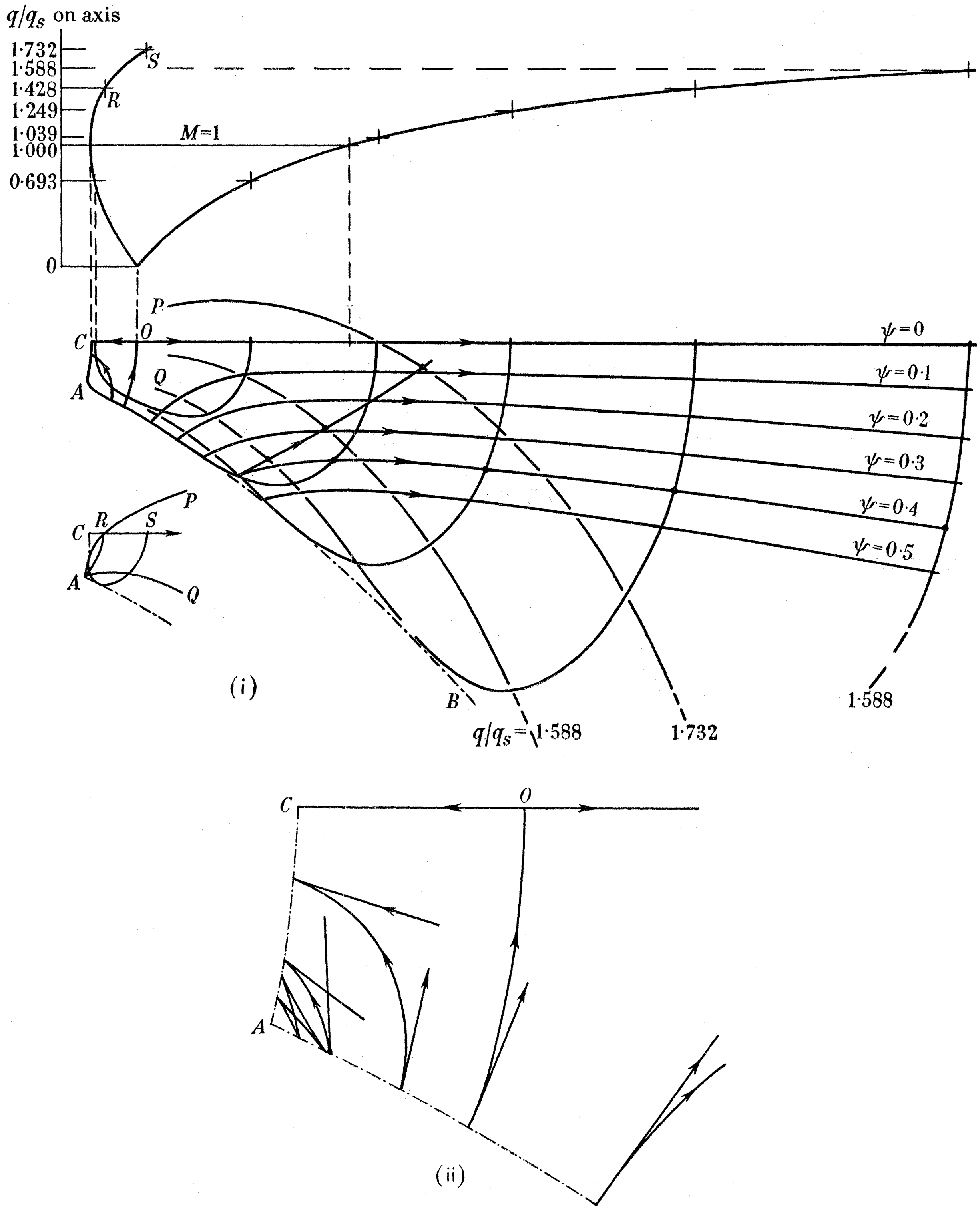

FIGURE 6. Trans-sonic nozzle flow, solution $\mathscr{R}\Omega_2$. (i) shows representative streamlines (arrowed) and isovels in the principal flow-sheet, and also continuations, across the limit-line *AB* into a second flow-sheet, of one streamline and of the supersonic isovels. (The left-hand part of the isovel $q/q_s = 1·039$ is indistinguishable, except in the corner near *A*, from the limit-lines *AB*, *AC*.) The repetition in the left lower corner indicates how the higher isovels enter the third and fourth flow-sheets near *A*. The upper curve shows the axial speed. (ii) shows, enlarged, the streamlines near *A*. In the four-sheeted flow-plane they are closed curves.

transformation (2·1) into functions of q, θ with simple branch points; and by (2·5) the branch point will be at $\theta = 0$ and $q = q_\infty < q_s$ provided κ is real, with

$$|\kappa| = \operatorname{arc\,tanh}\sqrt{\left(\frac{1-q_\infty^2/q^2}{1-q_\infty^2}\right)} - \frac{1}{q_s}\operatorname{arc\,tanh}\sqrt{\left(\frac{q_s^2-q_\infty^2}{1-q_\infty^2}\right)}. \tag{6·1}$$

The additional condition as to the infinity of the derivatives of Ω is met by choosing the function Ω_1 given by (4·49) and (4·50) with $p = 1$; and to get a real symmetrical solution we take minus the real part. Thus we may expect to get a family of aerofoil-type flows from the potential:

$$\Omega = \frac{1}{2(1+\alpha)}\int_0^{q^2}\frac{(1-\tau)^{-\beta}-1}{\tau}\,\mathrm{d}\tau + \mathscr{R}\left[\log(1-q\,\mathrm{e}^{-\mathrm{i}\phi}) - \sum_1^\infty q^n g_n^{(1)}(q^2)\,\mathrm{e}^{\mathrm{i}n\phi}\right], \tag{6·2}$$

where ϕ is related to θ by (2·1), with a real non-zero value of the parameter κ, related by (6·1) to the free-stream velocity q_∞; as $|\kappa|$ runs from 0 to ∞, q_∞ runs from q_s to 0. To find the corresponding position co-ordinates and stream function we put $p = 1$ in (4·54); writing

$$\Omega_p^+ = \sum_0^\infty q^n g_n^{(p)}(q^2)\,\mathrm{e}^{\mathrm{i}n\phi}, \tag{6·3}$$

and taking account of (4·20), (4·50) and (4·51) this gives

$$X = \mathscr{R}\left[\frac{2\mathrm{i}\alpha(1-q^2)^{1-\frac{1}{2}\beta}\sin\phi}{(1+\alpha)(1-2q\cos\phi+q^2)D} - \frac{\mathrm{e}^{-\mathrm{i}\phi}}{1-q\,\mathrm{e}^{-\mathrm{i}\phi}} + \frac{2\mathrm{i}\alpha\Omega_0^+\sin\phi}{1-2q\cos\phi+q^2} - \frac{\partial\Omega_1^+}{\partial q}\right], \tag{6·4}$$

$$Y = -\mathscr{I}\left[\frac{(1-q^2)^{1-\frac{1}{2}\beta}}{(1+\alpha)\,qD} + \frac{\Omega_0^+}{q}\right], \tag{6·5}$$

$$\begin{aligned}\psi = \mathscr{I}\Bigg[&\log(1-q\,\mathrm{e}^{-\mathrm{i}\phi}) + \frac{(1-q^2)^\beta q\,\mathrm{e}^{-\mathrm{i}\phi}}{1-q\,\mathrm{e}^{-\mathrm{i}\phi}}\left(\alpha\log(1-q^2) - \frac{1}{2}\int_0^{q^2}\frac{(1-\tau)^{-\beta}-1}{\tau}\,\mathrm{d}\tau\right)\\ &+\frac{(1-q^2)^{1+\frac{1}{2}\beta}}{(1+\alpha)D} - (1-q^2)^\beta\left(\frac{\alpha\log(1-q^2)}{1-q\,\mathrm{e}^{\mathrm{i}\phi}} + q\frac{\partial\Omega_2^+}{\partial q} - \frac{2\mathrm{i}\alpha q\Omega_1^+\sin\phi}{1-2q\cos\phi+q^2} - \Omega_0^+\right)\Bigg],\end{aligned} \tag{6·6}$$

with
$$D = 1-2(1+\alpha)\,q\cos\phi + (1+2\alpha)\,q^2. \tag{6·7}$$

And, to complete the statement of formulae, we are to put $\phi = \phi' + \mathrm{i}\zeta$; then ζ, θ are connected with q, ϕ' by (2·10) and (2·11), and finally (1·4) give x, y.

6·1. *The flow-field.* In the first place it is seen that the point at infinity in the *xy*- or flow-plane given by (6·4) and (6·5) corresponds to the hodograph point where $D = 0$, and so, by § 2, the relevant values of ϕ have their imaginary parts the same sign as κ. If this sign is positive the series (6·3) are rapidly convergent, and are of subsidiary effect in (6·4) to (6·6); but the contrary case is more complicated. Confining attention then to the simpler case, we shall show that:

When κ is positive and not too small (i.e. *when q_∞ is not too large*), *a bounded part, surrounding A, of the hodograph surface of figure 2 is in one-one correspondence with the part of the flow-plane outside a closed streamline $\psi = 0$. For all admissible values of κ this streamline is an aerofoil shape with cusped trailing edge and* (*at least for γ near* 1·4) *a blunt leading-edge.*

The proof is, in outline, as follows:

(i) Since D has a simple zero at the hodograph point A, a neighbourhood of this point has a one-one map on the far part of the flow-plane.

(ii) On the inner generator BAC of the hodograph surface ϕ is pure-imaginary, so Ω_p^+ and D are real. Thence $Y = \psi = \theta = 0$, so this generator maps on to the axis $y = 0$ of the flow-plane, with $x = X(q)$. As (q, ϕ) moves up the generator through A, x decreases to $-\infty$ and then starts decreasing from $+\infty$; and $q(x) < q_A = q_\infty$. Now as (q, ϕ) moves from A to C, $x(q)$ remains regular and either (a) $\mathrm{d}x/\mathrm{d}q$ remains always positive, so that AC has a one-one map on a ray $x_C < x < +\infty$ terminating in a stagnation point x_C; or (b) $\mathrm{d}x/\mathrm{d}q$ becomes zero at some point U between A and C, and the mapping becomes singular at U. In this case (see Appendix I) the locus $\psi = 0$ on the hodograph surface consists of the generator AC together with a curve intersecting it perpendicularly at U, and a neighbourhood of AU terminated by this curve maps uniquely on to a part of the xy-plane bounded by two arcs which meet tangentially so as to form a cusp. (It is because $q > q_U$ on AU that the map is unique.)

Of these two possibilities, the second actually occurs. For near C, where $q \sim 0$ and by (2·12) $q\,\mathrm{e}^{-\mathrm{i}\phi} \sim 1$, (6·2) gives the principal part $\Omega(q, \phi) \sim \mathscr{R} \log(1 - q\,\mathrm{e}^{-\mathrm{i}\phi})$, so from (2·17), $\Omega(q, \theta) \sim \alpha^{-1}(\log q + \kappa)$ and $x = \partial\Omega(q, \theta)/\partial q \sim (\alpha q)^{-1}$, which is large positive; thus x is large positive at both ends of AC, and there must be an intermediate turning point.

As regards the mapping of the hodograph arc AB, there are the same two possibilities. On this arc we have from (4·48)

$$\frac{\partial x}{\partial q} = \frac{\partial^2 \Omega(q, \theta)}{\partial q^2} = -\sum_2^\infty \frac{\Gamma(n\alpha + n)}{\Gamma(n\alpha + 1)\,\Gamma(n+1)} \left(\mathrm{e}^{n\kappa} - \frac{h_n\,\mathrm{e}^{-n\kappa}}{h_n^*}\right) \frac{\mathrm{d}^2}{\mathrm{d}q^2}\{q^n F_n(q^2)\}.$$

From (4·5) and (4·17) it may be shown that $h_2/h_2^* = \frac{3}{2}$ and that h_3/h_3^*, ... are between $\frac{3}{2}$ and 1, while $\mathrm{d}^2(q^n F_n)/\mathrm{d}q^2 > 0$ for $0 < q < q_s$. Hence if $\mathrm{e}^{4\kappa} > \frac{3}{2}$ we get case (a), and if $\mathrm{e}^{4\kappa} < \frac{3}{2}$ case (b). However, when γ is in the neighbourhood of 1·4 the case $\mathrm{e}^{4\kappa} < \frac{3}{2}$ is on other grounds (see the footnote to (iv) below) inadmissible, so we are left with case (a): the hodograph arc maps uniquely on to a ray $y = 0$, $-\infty < x < x_B$, and at x_B there is a regular stagnation point which is the leading edge of a blunt-nosed aerofoil $\psi = 0$.

(iii) In the limiting case of slow motion, $q_\infty \sim 0$ or $\kappa \sim \infty$, the formulae reduce to closed forms and a complete analytical discussion is relatively easy. On the hodograph surface, $\mathscr{I}\phi > \kappa/(1+\alpha)$, so for $\kappa \sim \infty$, $\mathscr{I}\phi$ is large, and the limiting formulae are obtained by putting

$$q\,\mathrm{e}^{-\mathrm{i}\phi} = \eta$$

and taking $q \sim 0$ with η remaining non-zero. This gives

for (2·7): $$q\,\mathrm{e}^{\kappa - \mathrm{i}\theta} = \eta(1-\eta)^\alpha,$$

for (6·4), (6·5): $$q(X + \mathrm{i}Y) = -\eta\{1 - (1+\alpha)\,\eta\}^{-1},$$

for (1·4): $$z = x + \mathrm{i}y = aq(X + \mathrm{i}Y)\,q^{-1}\,\mathrm{e}^{\mathrm{i}\theta} = -\frac{a\,\mathrm{e}^\kappa}{(1-\eta)^\alpha\{1 - (1+\alpha)\,\eta\}},$$

for (6·6): $$\psi = \mathscr{I}\left\{\log(1-\eta) + \frac{\eta}{1 - (1+\alpha)\,\eta}\right\},$$

and by taking $a = \mathrm{e}^{-\kappa}$ we get a finite limiting form for z. In the η-plane, the curve $\psi = 0$ consists of the real axis together with an oval intersecting it at $\eta = 0$, $(1+2\alpha)(1+\alpha)^{-2}$; and the interior of this oval maps uniquely on to the part of the z-plane outside the profile shown (for $\gamma = 1{\cdot}4$, $\alpha = 0{\cdot}724745$) in figure 7.

(iv) An argument from continuity now shows that when q_∞ is sufficiently small the circumstances are qualitatively the same as for $q_\infty \sim 0$: on the hodograph surface the streamline $\psi = 0$ consists of the generator BAC together with an oval cutting it at B and U (see figure 2), and the interior of this oval maps uniquely on to the flow-plane. From analogy with other hodograph investigations we must expect that, as q_∞ increases, the uniqueness of the mapping will break down through the appearance of limit-lines intersecting the streamline $\psi = 0$, and this is indeed the case. The greatest admissible value of q_∞ can be found only by

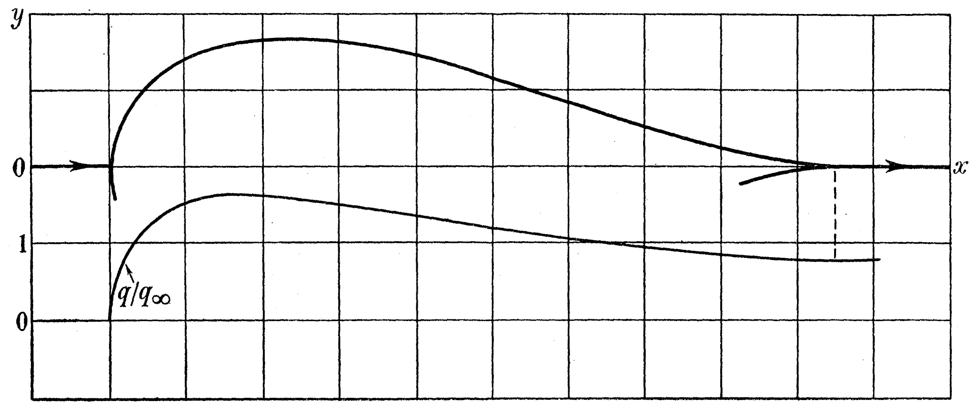

FIGURE 7. The limiting case $q_\infty \sim 0$: half of the aerofoil profile, and the speed q on it.

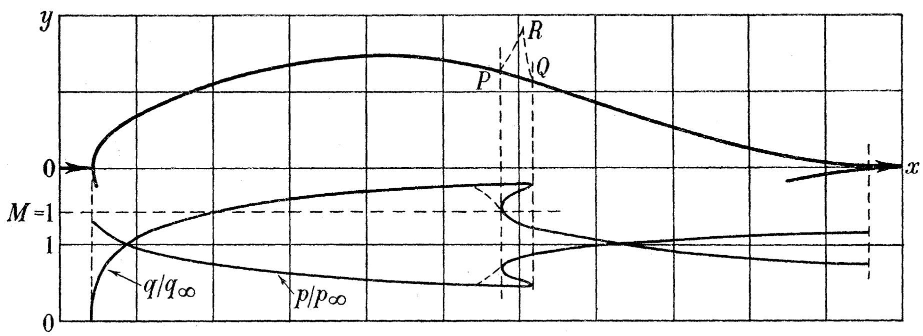

FIGURE 8. The streamline $\psi = 0$, with speed and pressure on it, for $\gamma = 1{\cdot}4$ and $M_\infty = 0{\cdot}660$.

arithmetical trial. In the case $\gamma = 1{\cdot}4$ the streamline $\psi = 0$ has been computed for $\kappa = 0{\cdot}2$† (corresponding to $q_\infty = 0{\cdot}283$, or Mach number $M_\infty = 0{\cdot}660$) and is shown in figure 8; it has two cusps P, Q on each flank‡, and there is a small adjacent part PQR of the exterior region where the mapping from the hodograph surface is 3 : 1 instead of 1 : 1. For a small decrease in q_∞ the 3 : 1 region becomes retracted within the curve $\psi = 0$; and, pending further calculations, the graphs in figure 8, with the kinks rounded off as indicated by dotted lines, may be taken as fair approximations to the case where M_∞ is about 0·6. On the flank of the aerofoil there is a limited region of supersonic flow, with the maximum speed occurring aft of the point where y is greatest; this feature is consonant with the commonsense suggestion that, by the decrease in y, the streamlines are here being 'encouraged' to spread out, which for locally supersonic flow implies increase in speed.

† This is greater than the critical value $\kappa = \frac{1}{4} \log \frac{3}{2} = 0{\cdot}1014$ noted in (ii) above.
‡ These cusps are masked in figure 8 by the thickness of the line.

6·2. *The limiting case* $q_\infty \sim q_s$, $\kappa \sim 0$. Here, on account of the cusps on its flanks, the curve $\psi = 0$ is no longer an aerofoil, and the flow so loses its chief physical interest. It is, however, of considerable mathematical interest since, as we shall show, *by choosing in* (1·4) $a = \kappa^{\frac{1}{2}}$ *we obtain in the limit* $q_\infty = q_s$, $\kappa = 0$ *a field consisting of a number of distinct Prandtl-Meyer flows, such as normally are 'missed' by a hodograph investigation.* In calling these flows 'distinct' we speak in the analytical sense; but where the xy-regions containing two of them abut, they join with continuity in q, θ but discontinuity in the higher derivatives. Together they form the type of solution commonly assumed for supersonic problems, and this is here obtained as the limit of a solution which is everywhere analytic.

The mechanism of this degeneracy is, in general and rather vague terms, as follows. For a Prandtl-Meyer flow, θ is functionally related to q; in fact the hodograph point (q, θ) is confined to a single characteristic. Now the critical locus $D = 0$ corresponds to a pair of characteristics, and the hodograph surface of figure 2 has, for $\kappa \sim 0$, a narrow band which is practically a cylinder standing on this critical locus. As $\kappa \to 0$ the breadth of this band tends to zero, and if, for some hodograph solution, its map on the flow-plane remains two-dimensional in the limit, a pair of Prandtl-Meyer flows will be obtained. The solution specified by (6·4) and (6·5) has the desired property because of the presence of D in the denominators of the leading terms; in the limit, $D \to 0$, but we preserve finite limits for x, y by taking the scale constant a in (1·4) equal to $\kappa^{\frac{1}{2}}$. Moreover, in a neighbourhood of the point A of figure 2, whose dimensions are small with κ, the real and unreal parts of D are independently small, and on this account its map remains two dimensional as $\kappa \to 0$. In the limit, therefore, we get a proper region of the flow plane corresponding to a single hodograph point, that is, a uniform stream; this is in addition to the two Prandtl-Meyer flows previously mentioned.

The details of the matter are as follows: The case to be considered is $\kappa \sim 0$, in (1·4) we take $a = \kappa^{\frac{1}{2}}$, and we exclude q from a neighbourhood of 1. Then the formulae (1·4), (6·4), (6·5) for our solution are equivalent to

$$x = X' \cos\theta - Y' \sin\theta, \quad y = X' \sin\theta + Y' \cos\theta, \tag{6·8}$$

$$X' = \mathscr{R}\left[\frac{2i\alpha(1-q^2)^{1-\frac{1}{2}\beta} \sin\phi}{(1+\alpha)(1-2q\cos\phi+q^2)\,\kappa^{-\frac{1}{2}}D} + O(\kappa^{\frac{1}{2}})\right], \tag{6·9}$$

$$Y' = -\mathscr{I}\left[\frac{(1-q^2)^{1-\frac{1}{2}\beta}}{(1+\alpha)\,q\kappa^{-\frac{1}{2}}D} + O(\kappa^{\frac{1}{2}})\right]. \tag{6·10}$$

Writing $\phi = \phi' + i\zeta$, we confine attention to the strip of the hodograph surface, i.e. the horn in figure 2, for which

$$\zeta = O(\kappa^{\delta}), \quad \kappa/\zeta = O(\kappa^{2\delta}), \tag{6·11}$$

where δ is a constant greater than $\frac{1}{8}$ and not exceeding $\frac{1}{3}$. Then we have approximations such as

$$\sinh 2\lambda(\zeta-\kappa) = 2\lambda\zeta\{1-\kappa/\zeta+\tfrac{2}{3}\lambda^2\zeta^2+O(\kappa^{4\delta})\},$$

and the equation (2·10) of the hodograph surface becomes

$$\begin{aligned} 2(1+\alpha)\,q\cos\phi' &= 1+(1+2\alpha)\,q^2-\frac{\alpha(1-q^2)\,\kappa}{(1+\alpha)\,\zeta}+\frac{\zeta^2}{6\alpha}\{(2\alpha+1)(3\alpha+2)\,q^2-(\alpha+2)\}+O(\kappa^{4\delta}) \\ &= 1+(1+2\alpha)\,q^2+O(\kappa^{2\delta}). \end{aligned} \tag{6·12}$$

Thence (6·7) gives

$$\kappa^{-\frac{1}{2}}D = \kappa^{-\frac{1}{2}}\{1-2(1+\alpha)\,q\cos\phi'\cosh\zeta+(1+2\alpha)\,q^2+2\mathrm{i}(1+\alpha)\,q\sin\phi'\sinh\zeta\}$$

$$= \frac{\alpha(1-q^2)\,\kappa^{\frac{1}{2}}}{(1+\alpha)\,\zeta}+\frac{\zeta^2}{3\alpha\kappa^{\frac{1}{2}}}\{1-\alpha-(1+2\alpha)\,(1+3\alpha)\,q^2\}+O(\kappa^{4\delta-\frac{1}{2}})$$
$$+2\mathrm{i}(1+\alpha)\,q\zeta\kappa^{-\frac{1}{2}}\sin\phi'\{1+O(\kappa^{2\delta})\}, \qquad (6\cdot13)$$

and (2·11) gives
$$\theta = \phi'-2\alpha\arctan\frac{q\sin\phi'}{1-q\cos\phi'}+O(\kappa^{2\delta}). \qquad (6\cdot14)$$

(i) First, put $\xi = \kappa^{-\frac{1}{2}}\zeta$, and let $\kappa\to 0$ with ξ fixed at any positive value; this is consistent with (6·11) provided $\delta\leqslant\frac{1}{4}$. Also let ϕ' be fixed at any value whose sine is not zero. Then $\zeta\to 0$, $\sin\phi\to\sin\phi'$, and by (6·12), $(1+\alpha)\,(1-2q\cos\phi+q^2)\to\alpha(1-q^2)$; while since $\delta>\frac{1}{8}$, (6·13) gives

$$\kappa^{-\frac{1}{2}}D\to\frac{\alpha(1-q^2)}{(1+\alpha)\,\xi}+2\mathrm{i}(1+\alpha)\,q\xi\sin\phi'.$$

Hence the solution (6·9), (6·10) takes the limiting form

$$\left.\begin{aligned} X' &= \frac{4(1+\alpha)^3\,q(1-q^2)^{-\frac{1}{2}\beta}\sin^2\phi'}{\alpha^2(1-q^2)^2\,\xi^{-3}+4(1+\alpha)^4\,q^2\xi\sin^2\phi'},\\ Y' &= \frac{2(1+\alpha)^2\,(1-q^2)^{1-\frac{1}{2}\beta}\sin\phi'}{\alpha^2(1-q^2)^2\,\xi^{-3}+4(1+\alpha)^4\,q^2\xi\sin^2\phi'},\end{aligned}\right\} \qquad (6\cdot15)$$

while in (6·12) and (6·14) the $O(\kappa^{2\delta})$ terms disappear so that ϕ', θ become functions of q only; and thence x, y follow from (6·8).

Since θ is a function of q only, while x, y depend also on the parameter ξ, these formulae represent a pair of Prandtl-Meyer flows, one for $\phi'>0$ and the other for $\phi'<0$. It is easily calculated that, for the flow with $\phi'>0$, the streamlines lie as in the part of figure 9 to the right of the radii Oy, OQ. HK is a limit line, which is present because the denominator in (6·15) takes a minimum value for

$$\xi = 3^{\frac{1}{4}}\{(1-q^2)\,\alpha\,\mathrm{cosec}\,\phi'/2(1+\alpha)^2\,q\}^{\frac{1}{2}} = \xi_0(q); \qquad (6\cdot16)$$

and (6·15) give each point (x, y) doubly, once for $\xi<\xi_0$ and once for $\xi>\xi_0$. Since, however, the denominator is analytically equivalent to a single parameter, this limit-line is a removable singularity, and the Prandtl-Meyer flow *could* continue regularly across it.

(ii) Secondly, let us confine attention to a neighbourhood of the hodograph point A, by putting
$$\zeta = C^{\frac{1}{3}}\kappa^{\frac{1}{3}}(1-\kappa^{\frac{1}{6}}r), \quad \phi' = \kappa^{\frac{5}{6}}s, \qquad (6\cdot17)$$

where
$$C = 6\alpha^2(1+\alpha)^{-1}\,(1+2\alpha)^{-1}, \qquad (6\cdot18)$$

and r, s are parameters to be held fixed as $\kappa\to 0$. This is consistent with (6·11), where we can take $\delta=\frac{1}{3}$; (6·12) gives

$$q = (1+2\alpha)^{-1}+O(\kappa^{\frac{2}{3}}) = q_s+O(\kappa^{\frac{2}{3}}), \qquad (6\cdot19)$$

and (6·13) becomes

$$\kappa^{-\frac{1}{2}}D = \frac{2(1+\alpha)}{1+2\alpha}\,[C^{\frac{2}{3}}\kappa^{\frac{1}{3}}r\{1+O(\kappa^{\frac{1}{6}})\}+\mathrm{i}C^{\frac{1}{3}}\kappa^{\frac{2}{3}}s\{1+O(\kappa^{\frac{1}{6}})\}],$$

where the real term would have been of order $\kappa^{\frac{1}{6}}$ but for the special choice (6·18) of C. Also

$$\sin\phi = \mathrm{i}\cos\phi'\sinh\zeta\{1+O(\phi'/\zeta)\} = \mathrm{i}C^{\frac{1}{3}}\kappa^{\frac{1}{3}}\{1+O(\kappa^{\frac{1}{2}})\},$$

and for $\kappa \to 0$ the solution (6·9) and (6·10) takes the limiting form

$$X' = -\frac{(1-q_s^2)^{-\frac{1}{2}\beta}}{(1+q_s)\,C^{\frac{1}{2}}\,r}, \quad Y' = \frac{2(1-q_s^2)^{1-\frac{1}{2}\beta}\,s}{(1+q_s)^2\,Cr^2}. \tag{6·20}$$

The associated limits of q, θ are q_s, 0, so since r, s are unrestricted these formulae represent two uniform streams, parallel to Ox and of speed q_s, one for $r>0$ covering the half-plane $x<0$, and the other for $r<0$ covering the half-plane $x>0$. Figure 9 shows only the upper half of the first one.

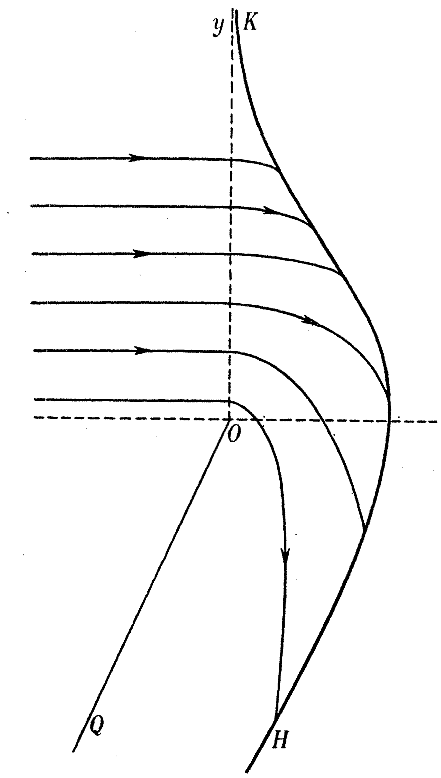

FIGURE 9. The limit-case of Prandtl-Meyer flows.

The formulae (6·15) are of course analytically distinct from (6·20). This is due to non-uniformity in the limiting processes, in the former case for ϕ' near 0 and in the latter for r near ∞.

(iii) Implicit in the preceding are conclusions concerning the case where κ is positive and small; but it is to be noted that the non-axial part of the streamline $\psi = 0$, which was the primary object of investigation in § 6·1, escapes that of § 6·2 since it does not lie in the strip (6·11) of the hodograph surface.† As κ decreases the limit-lines RP, RQ of figure 8 grow upwards, and it may be proved that, for $\kappa \sim 0$, their common point R is at $x = A\kappa^{\frac{1}{3}}$, $y = B\kappa^{-\frac{1}{3}}$, where (for $\gamma = 1·4$) $A = 0·33$, $B = 0·36$ approximately. In the limit $\kappa = 0$ they become Oy, HK in figure 9. For $\kappa \neq 0$ the remote part of the flow proceeds, above R, regularly from $x = -\infty$ to $x = +\infty$. The nearer part falls approximately into four sections, of which

† This strip does not completely girdle the hodograph surface since a neighbourhood of the outer generator where $\phi' \sim 0$ and $q \sim 1$ is to be excluded. The streamline $\psi = 0$, by twice traversing this neighbourhood, encloses the strip.

the first and the fourth are specified approximately by (6·20) with $r \lessgtr 0$, and the second and third by (6·15) with $\xi \lessgtr \xi_0$. The junction between first and second is regular, but the junctions of the third with the second and fourth are across the limit-lines HK, Oy.

6·3. *Other flow-cases.* (i) The potential $\Omega = \mathscr{R}\Omega_2(q,\phi)$, taken with $\kappa \neq 0$ in (2·1), gives a nozzle flow in which the axial speed is everywhere subsonic, with its maximum at the throat.

(ii) The potential $\Omega = \mathscr{R}(\Omega_1 + C\Omega_2)$ gives a stream-flow past an aerofoil with cusped trailing edge, whose thickness-ratio can be adjusted by choice of the constant C.

Appendix 1. Cusped streamlines in a subsonic region†

We shall confine attention to fields of flow having an axis of symmetry, such as are specified by a potential $\Omega(q,\theta)$ which is even in θ. The case to be investigated is that in which $\partial^2\Omega/\partial q^2$ vanishes at a point $q = q_0$, $\theta = 0$, where $0 < q_0 < q_s$. Near this point let

$$\Omega = a(q-q_0) + \tfrac{1}{2}b\theta^2 + \tfrac{1}{6}c(q-q_0)^3 + \tfrac{1}{2}d(q-q_0)\,\theta^2 + \tfrac{1}{24}e(q-q_0)^4 + \tfrac{1}{4}f(q-q_0)^2\,\theta^2 + \tfrac{1}{24}g\theta^4 + \dots.$$

Substituting this series into the hodograph equation (1·1) we find the following relations between the coefficients:

$$aq_0 + b = 0, \quad (a+d)\,(1-q_0^2/q_s^2) + cq_0^2(1-q_0^2) = 0,$$
$$(g+dq_0)\,(1-q_0^2/q_s^2) + fq_0^2(1-q_0^2) = 0,$$

and by these we can eliminate b, d, g from all subsequent formulae.

From (1·3), (1·4) and (1·5) there follow the developments for the position co-ordinates and stream function:

$$x - a = \tfrac{1}{2}c(q-q_0)^2 - \frac{1}{2}\frac{cq_0^2(1-q_0^2)\,\theta^2}{1-q_0^2/q_s^2} + \dots,$$

$$y = -\frac{cq_0(1-q_0^2)}{1-q_0^2/q_s^2}\left\{(q-q_0)\,\theta + \left(\frac{f}{6c} + \frac{q_0}{3}\right)\theta^3\right\} + \left\{\tfrac{1}{2}c + \frac{c(1-q_0^2)}{1-q_0^2/q_s^2} + \frac{f}{2q_0}\right\}(q-q_0)^2\,\theta + \dots,$$

$$q^{-1}(1-q^2)^{-\beta}\,\psi = \frac{cq_0(1-q_0^2)}{1-q_0^2/q_s^2}\left\{(q-q_0)\,\theta + \frac{f}{6c}\theta^3\right\} - \left\{\tfrac{1}{2}c + \frac{c(1-q_0^2)}{1-q_0^2/q_s^2} + \frac{f}{2q_0}\right\}(q-q_0)^2\,\theta + \dots.$$

Confining attention to the case $c \neq 0$, it follows that each of the loci $y = 0$, $\psi = 0$ in the hodograph plane consists of the axis $\theta = 0$ together with a curve, approximately parabolic, intersecting it perpendicularly where $q = q_0$ (figure 10); and since $q_0 > 0$ *the y-locus is to the left of the ψ-locus*. Also, since $0 < q_0 < q_s$, the locus $x - a = 0$ consists of two curves interlaced with the two branches of the locus $y = 0$. Hence it follows that as (q,θ) circles once round $(q_0, 0)$, the point (x,y) circles twice round $(a,0)$. The parabolic part of the locus $\psi = 0$ divides the hodograph plane into two parts; the one on the right is in one-one correspondence with a part of the xy-plane bounded by two curves forming a cusp, given parametrically by

$$x - a = -\frac{cq_0^2(1-q_0^2)}{2(1-q_0^2/q_s^2)}\,\theta^2 + \dots, \quad y = -\frac{cq_0^2(1-q_0^2)}{3(1-q_0^2/q_s^2)}\,\theta^3 + \dots;$$

while for the one on the left the corresponding xy-region is a Riemann surface with an overlap across the cusp (figure 10).

† The singularity here treated has been noticed briefly by Craggs (1948). The point in what follows is to find a criterion to distinguish between the cases (ii) and (iii) of figure 10; Craggs does not deal with this.

The first of these cases gives a possible physical flow; the second, on account of the overlap, does not. It is seen that the half of the hodograph plane which gives the physically acceptable flow is the one that contains the half-axis $\theta = 0$, $q > q_0$. Hence the conclusion:

For the flow defined by a potential $\Omega(q, \theta)$ which is even in θ let $\partial^2\Omega/\partial q^2$ vanish at a point $q = q_0$, $\theta = 0$, where $0 < q_0 < q_s$, with $\partial^3\Omega/\partial q^3 \neq 0$. Then the streamline $\psi = 0$ divides the hodograph plane near $(q_0, 0)$ into four compartments, and the pair of these which abut along the ray $\theta = 0$, $q > q_0$ is in one-one correspondence with a part of the xy-plane bounded by two arcs which together form a cusp, as in figure 10.

The distinction between the cases $f/c > 0$, $f/c < 0$ is that in the former, $q < q_0$ on the arcs forming the cusp, whereas in the latter $q > q_0$; in both cases $q > q_0$ on the straight streamline which prolongs the cusp.

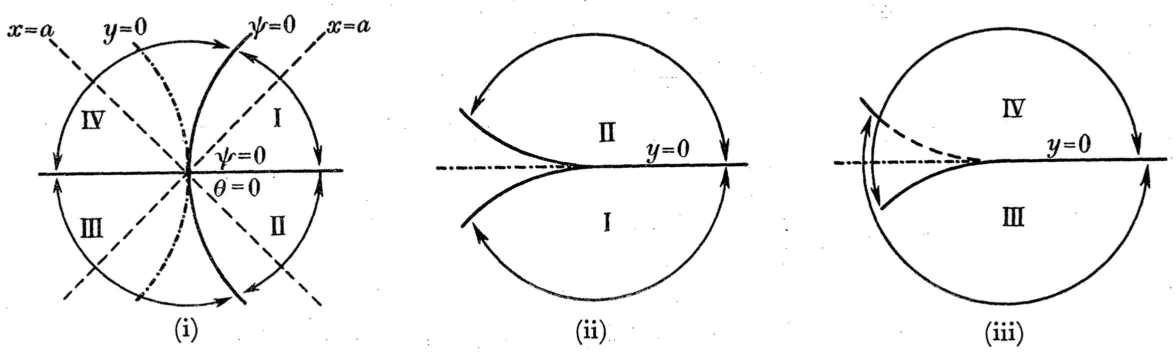

FIGURE 10. (i) hodograph plane; (ii) and (iii) flow plane. (Full line: $\psi = 0$. Chain dot: $y = 0$.)

APPENDIX 2. ANALOGIES FOR THE HODOGRAPH THEORY

The key facts of this paper are (i) that the transformation (2·1) has a branch locus which is characteristic for the hodograph equation, and (ii) that the function Ω_0 defined in (4·19) has an analytic continuation which is single valued for $0 \leqslant q < 1$ and all real ϕ. Neither of these properties is a priori at all evident, so it will be of interest to indicate briefly the considerations leading to the choice of the formulae (2·1) and (4·19) as starting-points.

The leading idea is that the attack on the hodograph equation should be guided by the prior study of simpler but analogous equations. One such equation is obtained from the hodograph equation (1·1) by putting

$$2\alpha q = t \tag{A 1}$$

and letting α become infinite, with t held fixed. This gives

$$t^2\Omega_{tt} + t\Omega_t + \Omega_{\theta\theta} - t^2(t\Omega_t + \Omega_{\theta\theta}) = 0. \tag{A 2}$$

If we omit the term $t^3\Omega_t$ (which does not alter the characteristics), the resulting equation is satisfied by a general Kapteyn series $\Omega = \Sigma A_\nu \mathrm{e}^{\mathrm{i}\nu\theta} J_\nu(\nu t)$; and the well-known formula (valid for $0 \leqslant t < 1$)

$$\sum_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}n\theta} J_n(nt) = (1 - t\cos\phi)^{-1}, \tag{A 3}$$

where

$$\theta = \phi - t\sin\phi, \tag{A 4}$$

suggests that we study the transformation (A 4) in relation to the equation (A 2). It is immediately found that its branch locus is characteristic for the equation.

The suggestion therefore is to find a generalization of (A 4), and it does not require much experimenting to hit on the replacement of $t\sin\phi$ by an expression having this as its limit, viz.

$$\mathrm{i}\alpha\log\left(1-\frac{t\,\mathrm{e}^{\mathrm{i}\phi}}{2\alpha}\right)-\mathrm{i}\alpha\log\left(1-\frac{t\,\mathrm{e}^{-\mathrm{i}\phi}}{2\alpha}\right).$$

Putting $t = 2\alpha q$ as in (A 1) we are led to (2·1).

Next, from (2·16) it is seen that the hypergeometric function $F(\nu\alpha+\nu, -\nu\alpha; \nu+1; q^2)$ plays the same part in relation to our transformation (2·1) as $J_\nu(\nu t)$ does in relation to Kepler's equation (A 4). On account of (4·6), this function is 'close' to the $F(a_\nu, b_\nu; \nu+1; q^2)$ which belongs to the hodograph equation, and it is analytically simpler because it lacks the branch points which affect a_ν, b_ν as functions of ν. The differential equation having the elementary solutions $\mathrm{e}^{\pm\mathrm{i}\nu\theta}\, q^\nu F(\nu\alpha+\nu, -\nu\alpha; \nu+1; q^2)$ is found to be

$$q^2(1-q^2)\,\Omega_{qq}+q\Omega_q+(1-q^2/q_s^2)\,\Omega_{\theta\theta}=0. \tag{A 5}$$

On account of (2·16) this has the solution $\Omega = \phi$, and by differentiation with respect to θ we obtain another closed solution

$$\Omega=\frac{\partial\phi}{\partial\theta}-\frac{1}{1+\alpha}=\frac{\alpha(1-q^2)}{(1+\alpha)\,D}$$
$$=\sum_{-\infty}^{\infty}\frac{\Gamma(n\alpha+n)}{\Gamma(n\alpha)\,\Gamma(n+1)}\,q^nF(n\alpha+n, -n\alpha; n+1; q^2)\,\mathrm{e}^{\mathrm{i}n\theta}, \tag{A 6}$$

where $D = 1-2(1+\alpha)\,q\cos\phi+(1+2\alpha)\,q^2$; when n is a negative integer, the coefficient of $\mathrm{e}^{\mathrm{i}n\theta}$ is to be interpreted as a limit, as in (4·11).

The series (4·19) defining Ω_0 is now suggested as an analogue of the one just written; from the expression

$$\frac{\Gamma(\nu\alpha+\nu)}{\Gamma(\nu\alpha+1)\,\Gamma(\nu)}\,q^\nu F(a_\nu, b_\nu; \nu+1; q^2)$$

we obtain the coefficient of $\mathrm{e}^{-\mathrm{i}n\theta}$ by putting $\nu = n$, and the coefficient of $\mathrm{e}^{\mathrm{i}n\theta}$ by taking the limit as $\nu\to -n$. One's initial hope that this series might be an elementary function of q, ϕ is not fulfilled; but the formula (4·20) found for Ω_0 seems to be, for the hodograph equation, the nearest attainable analogue to (A 6) or (A 3), and on this account I have called Ω_0 the principal solution. The branch points of a_ν, b_ν seem to be essential obstacles to the attainment of anything simpler.

It may be remarked that by putting in (A 4)

$$t = 1+\mu T, \quad \phi = (2\mu)^{\frac{1}{2}}\,\Phi, \quad \theta = 2^{\frac{1}{2}}\mu^{\frac{3}{2}}\Theta$$

and letting $\mu\to 0$ we obtain

$$\Theta = \tfrac{1}{3}\Phi^3 - T\Phi, \tag{A 7}$$

while (A 2) becomes the Tricomi equation $\Omega_{TT} = T\Omega_{\Theta\Theta}$; its transform by (A 7) is

$$(\Phi^2-T)\,\Omega_{TT}+2\Phi\Omega_{T\Phi}+\Omega_{\Phi\Phi}=0,$$

and solutions of this such as $\Omega = \Phi$ become branched functions of T, Θ.

Finally, it may be remarked that (A 5) can be interpreted as the exact hodograph equation of an 'ideal' gas for which the pressure-density relation does not follow the polytropic law, but is fairly close to it when the density is not too small (cf. von Mises & Schiffer (1948) and Tomotika & Tamada (1951) in relation to Tricomi's equation and Bessel's equation). In this paper, however, I have chosen to deal with the true hodograph equation of a polytropic gas.

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Table 1. The function $\Omega_0(q, \phi)$ for $\gamma=1\cdot4$, $\beta=2\cdot5$

$$\Omega_0=\frac{(1-q^2)^{-\frac{1}{2}\beta}}{(1+\alpha)\{1-2(1+\alpha)\,q\cos\phi+(1+2\alpha)\,q^2\}}-\frac{1}{1+\alpha}+\sum_{n=1}^{\infty}q^n g_n(q^2)\,\mathrm{e}^{in\phi},$$

where $\alpha=0\cdot72474487$. Writing $g_n(q^2)=a_{n0}+a_{n1}q^2+a_{n2}q^4+\dots$, the table gives the value of a_{nr}.

	$r=0$	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$
$n=1$	1·0	0·625	0·567308	0·557408	0·562944	0·574943
2	0·5	0·396725	0·350296	0·328427	0·317504	0·312303
3	0·254717	0·243631	0·214483	0·194462	0·180587	0·170617
4	0·143588	0·166272	0·149989	0·134785	0·122532	0·112705
5	0·087542	0·123248	0·115528	0·104691	0·094778	0·086227
6	0·056176	0·096589	0·094529	0·087131	0·079361	0·07220
7	0·037123	0·078619	0·080353	0·075591	0·06961	0·06371
8	0·024800	0·065722	0·070044	0·06730	0·06279	0·05796
9	0·016433	0·056024	0·06213	0·06095	0·05764	0·05373
10	0·010532	0·04847	0·05582	0·05585	0·05352	0·05038
11	0·00624	0·04241	0·05063	0·05162	0·05010	0·04762
12	0·00305	0·03745	0·04626	0·04802	0·04717	
13	0·00062	0·03332	0·04253	0·04489		
14	−0·00125	0·02982	0·03929			
15	−0·00272	0·02682				
16	−0·00388					

	$r=6$	$r=7$	$r=8$	$r=9$	$r=10$	$r=11$
$n=1$	0·589989	0·606534	0·623797	0·641357	0·65897	0·67651
2	0·310423	0·310613	0·312158	0·31462	0·31773	0·32129
3	0·163233	0·157638	0·15333	0·14997	0·14734	0·14527
4	0·104703	0·09807	0·09250	0·08774	0·08363	0·08004
5	0·07890	0·07258	0·06707	0·06224	0·05795	0·05411
6	0·06580	0·06012	0·05506	0·05053	0·04644	
7	0·05822	0·05321	0·04866	0·04453		
8	0·05329	0·04891	0·04486			
9	0·04976	0·04594				
10	0·04704					

	$r=12$	$r=13$	$r=14$	$r=15$
$n=1$	0·69387	0·71102	0·72793	0·74459
2	0·32517	0·32930	0·33359	
3	0·14365	0·14239		
4	0·07688			

Table 2. The function $\Omega_1(q, \phi)$ for $\gamma = 1{\cdot}4$, $\beta = 2{\cdot}5$

$$\Omega_1(q, \phi) = -\log(1 - q\,\mathrm{e}^{-\mathrm{i}\phi}) + \sum_{n=0}^{\infty} q^n g_n^{(1)}(q^2)\,\mathrm{e}^{\mathrm{i}n\phi}.$$

Writing $g_n^{(1)}(q^2) = b_{n0} + b_{n1}q^2 + b_{n2}q^4 + \ldots$, the table gives the value of b_{nr}.

	$r=0$	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$
$n=0$	0·0	−0·724745	−0·634152	−0·634152	−0·653969	−0·680128
1	−2·0	−0·987372	−0·910517	−0·906181	−0·924010	−0·950572
2	−0·387628	−0·241950	−0·236923	−0·234168	−0·234163	−0·235914
3	−0·055866	−0·041274	−0·060280	−0·065667	−0·067552	−0·068288
4	+0·032034	+0·022372	−0·003704	−0·013623	−0·018176	−0·020455
5	0·057649	0·046057	+0·019044	+0·007222	+0·001089	−0·002395
6	0·063843	0·055410	0·029665	0·017310	0·010402	+0·00619
7	0·063260	0·058786	0·035038	0·022825	0·01562	0·01102
8	0·060256	0·059414	0·037813	0·02607	0·01884	0·01406
9	0·056488	0·058707	0·03918	0·02804	0·02095	0·01613
10	0·052620	0·05734	0·03972	0·02925	0·02239	0·01761
11	0·04892	0·05566	0·03978	0·02997	0·02338	0·01869
12	0·04549	0·05385	0·03952	0·03037	0·02407	
13	0·04234	0·05201	0·03907	0·03054		
14	0·03949	0·05019	0·03850			
15	0·03689	0·04843				
16	0·03453					

	$r=6$	$r=7$	$r=8$	$r=9$	$r=10$	$r=11$
$n=0$	−0·708466	−0·737383	−0·766187	−0·794565	−0·822375	−0·84956
1	−0·980921	−1·012893	−1·045437	−1·078007	−1·11031	−1·14219
2	−0·238755	−0·242282	−0·246247	−0·25050	−0·25492	−0·25946
3	−0·068613	−0·068798	−0·06895	−0·06912	−0·06932	−0·06955
4	−0·021627	−0·02221	−0·02246	−0·02252	−0·02247	−0·02234
5	−0·00448	−0·00575	−0·00653	−0·00699	−0·00724	−0·00735
6	+0·00348	+0·00169	+0·00046	−0·00037	−0·00094	
7	0·00793	0·00577	0·00423	+0·00311		
8	0·01075	0·00836	0·00660			
9	0·01270	0·01017				
10	0·01413					

	$r=12$	$r=13$	$r=14$	$r=15$	$r=16$
$n=0$	−0·87611	−0·90203	−0·92734	−0·95207	−0·97624
1	−1·17356	−1·20439	−1·23465	−1·26435	
2	−0·26407	−0·26871	−0·27335		
3	−0·06982	−0·07012			
4	−0·02217				

Table 3. The function $\Omega_2(q, \phi)$ for $\gamma = 1{\cdot}4$, $\beta = 2{\cdot}5$

$$\Omega_2(q, \phi) = \sum_{n=-\infty}^{\infty} q^{|n|} g_n^{(2)}(q^2)\, e^{in\phi}.$$

Writing $g_n^{(2)}(q^2) = c_{n0} + c_{n1} q^2 + c_{n2} q^4 + \ldots$, the table gives $-c_{nr}$ for $n = 2, 3, 4, \ldots$ and c_{nr} for $n = 1, 0, -1, -2, \ldots$.

	$r=0$	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$
$n=2$	0·530931	0·157321	0·263446	0·313146	0·349455	0·380001
3	0·558185	0·208988	0·270184	0·300616	0·324890	0·346582
4	0·450736	0·175490	0·222290	0·244660	0·262482	0·278545
5	0·361069	0·140988	0·181648	0·200278	0·214687	0·227524
6	0·294959	0·113556	0·150798	0·167438	0·179878	0·19075
7	0·246129	0·092390	0·127277	0·142700	0·15390	0·16348
8	0·209257	0·075936	0·108954	0·12354	0·13388	0·14257
9	0·180735	0·062947	0·09436	0·10830	0·11802	0·12605
10	0·158181	0·05253	0·08251	0·09591	0·10515	0·11268
11	0·14000	0·04405	0·07272	0·08566	0·09450	0·10163
12	0·12509	0·03706	0·06452	0·07704	0·08554	
13	0·11269	0·03122	0·05757	0·06969		
14	0·10224	0·02630	0·05161			
15	0·09335	0·02212				
16	0·08569					

	$r=6$	$r=7$	$r=8$	$r=9$	$r=10$	$r=11$
$n=2$	0·407242	0·432256	0·455614	0·47766	0·49863	0·51868
3	0·366702	0·385668	0·40371	0·42096	0·43754	0·45352
4	0·293572	0·30784	0·32148	0·33459	0·34723	0·35945
5	0·23949	0·25084	0·26170	0·27214	0·28221	0·29196
6	0·20078	0·21025	0·21929	0·22797	0·23633	
7	0·17222	0·18041	0·18819	0·19564		
8	0·15039	0·15766	0·16453			
9	0·13320	0·13978				
10	0·11930					

	$r=12$	$r=13$	$r=14$
$n=2$	0·53794	0·55650	0·57444
3	0·46897	0·48392	
4	0·37128		

	$r=0$	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$
$n=1$	2·0	0·905931	0·597451	0·509759	0·475831	0·462267
0	0·0	2·174235	1·358897	1·240622	1·236690	1·266731
−1	1·0	1·612373	1·214541	1·154145	1·164167	1·197205
−2	−0·112372	0·866582	0·637468	0·595191	0·594163	0·607230
−3	−0·251261	0·597853	0·437058	0·404847	0·401860	0·409134
−4	−0·269675	0·457449	0·333833	0·307949	0·304631	0·309367
−5	−0·261977	0·370791	0·270518	0·248948	0·245718	0·249104
−6	−0·248028	0·311868	0·227588	0·209134	0·206107	0·20869
−7	−0·233253	0·269165	0·196514	0·180408	0·17760	0·17966
−8	−0·219270	0·236777	0·172956	0·15868	0·15609	0·15778
−9	−0·206516	0·211363	0·15447	0·14166	0·13926	0·14069
−10	−0·195028	0·19089	0·13957	0·12796	0·12573	0·12697
−11	−0·18471	0·17403	0·12731	0·11669	0·11462	0·11570
−12	−0·17544	0·15992	0·11703	0·10725	0·10532	
−13	−0·16709	0·14792	0·10829	0·09923		
−14	−0·15953	0·13760	0·10077			
−15	−0·15266	0·12863				
−16	−0·14640					

	$r=6$	$r=7$	$r=8$	$r=9$	$r=10$	$r=11$
$n=1$	0·458253	0·459322	0·463323	0·469109	0·47602	0·48366
0	1·309446	1·357246	1·406940	1·457041	1·506818	1·55591
−1	1·239180	1·284740	1·331544	1·378474	1·42497	1·47077
−2	0·626061	0·647403	0·669798	0·69253	0·71523	0·73771
−3	0·420729	0·434286	0·44873	0·46352	0·47837	0·49313
−4	0·317560	0·32736	0·33792	0·34880	0·35977	0·37071
−5	0·25537	0·26299	0·27126	0·27984	0·28851	0·29717
−6	0·21372	0·21993	0·22672	0·23377	0·24093	
−7	0·18385	0·18908	0·19482	0·20081		
−8	0·16137	0·16588	0·17085			
−9	0·14382	0·14778				
−10	0·12974					

	$r=12$	$r=13$	$r=14$	$r=15$	$r=16$
$n=1$	0·49176	0·50016	0·50875	0·51745	
0	1·60412	1·65138	1·69767	1·74299	1·78735
−1	1·51571	1·55975	1·60288	1·64510	
−2	0·75985	0·78160	0·80295		
−3	0·50771	0·52207			
−4	0·38154				

TABLE 4. STREAM FUNCTION AND POSITION CO-ORDINATES FOR SOLUTION $\mathscr{R}\Omega_2$

The functions tabulated are $10^4\psi$, 10^3X, -10^3Y, whence x, y follow from (1·4)

$10^4\psi(q^2, \phi)$

ϕ^0 \ q^2	0·02	0·04	0·06	0·08	0·10	0·12	0·14	0·16	0·18
0	0	0	0	0	0	0	0	0	0
1	14	31	51	72	96	122	151	181	214
2	28	62	101	144	192	244	301	362	427
3	42	93	151	216	288	366	450	541	639
4	55	123	201	288	383	487	599	720	850
5	69	154	251	359	477	606	746	897	1058
6	83	184	300	429	571	725	892	1071	1264
7	96	214	349	499	663	842	1035	1243	1467
8	110	244	397	567	754	957	1177	1413	1666
9	123	273	444	635	844	1071	1316	1579	1861
10	136	302	491	701	932	1182	1452	1742	2052
11	149	330	537	767	1018	1291	1585	1900	2238
12	161	358	582	831	1102	1397	1714	2055	2418
13	174	386	626	893	1185	1501	1841	2205	2593
14	186	412	669	954	1265	1601	1963	2350	2762
15	198	438	711	1013	1343	1699	2081	2490	2925
16	209	464	752	1071	1418	1793	2195	2625	3081
17	221	489	792	1127	1491	1884	2305	2754	3231
18	232	513	830	1181	1561	1972	2411	2878	3374
19	242	536	867	1232	1629	2055	2511	2996	3509
20	253	558	903	1282	1693	2135	2607	3108	3637
21	263	580	937	1330	1755	2212	2698	3214	3758
22	272	601	970	1375	1814	2284	2784	3313	3872
23	282	621	1001	1419	1870	2352	2865	3407	3978
24	290	640	1031	1460	1922	2416	2941	3494	4076
25	299	658	1059	1498	1971	2476	3011	3575	4167
26	307	675	1086	1535	2018	2532	3077	3650	4250
27	315	691	1111	1569	2061	2584	3137	3718	4326
28	322	706	1134	1600	2100	2631	3192	3780	4394
29	329	720	1156	1630	2137	2675	3242	3835	4454
30	335	733	1176	1656	2170	2714	3286	3885	4508
31	341	746	1195	1681	2200	2749	3326	3928	4554
32	346	757	1211	1703	2227	2780	3360	3965	4593
33	351	767	1226	1722	2250	2807	3389	3996	4625
34	356	776	1240	1739	2270	2829	3414	4021	4650
35	360	784	1251	1754	2287	2848	3433	4041	4668
36	364	791	1261	1766	2301	2863	3448	4054	4680
37	367	797	1269	1776	2312	2873	3458	4063	4686
38	370	802	1276	1783	2319	2880	3463	4065	4685
39	372	806	1281	1788	2324	2883	3464	4063	4679
40	374	809	1284	1791	2326	2883	3460	4056	4666
41	375	811	1286	1792	2324	2879	3453	4043	4648
42	376	811	1286	1790	2320	2871	3441	4026	4625
43	376	811	1284	1786	2313	2860	3425	4004	4596
44	376	810	1281	1781	2303	2846	3405	3978	4563
45	375	808	1276	1772	2291	2828	3381	3947	4524
46	374	805	1270	1762	2276	2808	3354	3913	4482
47	373	801	1263	1750	2259	2784	3323	3874	4434
48	371	796	1254	1736	2239	2757	3289	3832	4383
49	369	790	1243	1721	2217	2728	3252	3786	4328
50	366	783	1232	1703	2192	2696	3211	3736	4269
51	363	776	1219	1683	2165	2661	3168	3684	4206
52	359	767	1204	1662	2137	2624	3122	3628	4140
53	355	758	1189	1640	2106	2585	3073	3569	4071
54	351	748	1172	1615	2073	2543	3022	3508	3999

TABLE 4 (*cont.*)

$10^4\psi(q^2, \phi)$

ϕ^0 \ q^2	0·20	0·22	0·24	0·26	0·28	0·30	0·32	0·34
0	0	0	0	0	0	0	0	0
1	249	287	327	370	416	465	518	574
2	497	572	653	739	831	929	1034	1146
3	744	857	977	1105	1243	1389	1546	1713
4	989	1138	1298	1468	1650	1844	2052	2273
5	1232	1417	1615	1826	2052	2293	2549	2823
6	1471	1691	1927	2179	2447	2733	3037	3362
7	1706	1961	2234	2524	2834	3163	3514	3887
8	1937	2226	2534	2862	3211	3582	3977	4396
9	2163	2484	2827	3191	3578	3989	4425	4888
10	2383	2736	3111	3510	3933	4382	4858	5362
11	2597	2980	3387	3819	4276	4761	5273	5816
12	2805	3217	3654	4116	4606	5124	5671	6249
13	3006	3445	3910	4402	4922	5471	6050	6660
14	3200	3665	4157	4676	5224	5801	6409	7048
15	3387	3876	4392	4937	5511	6114	6748	7413
16	3565	4077	4616	5185	5782	6409	7067	7756
17	3735	4268	4829	5419	6038	6687	7365	8074
18	3897	4450	5030	5640	6278	6946	7643	8370
19	4051	4621	5219	5846	6502	7186	7899	8642
20	4195	4782	5396	6039	6710	7409	8136	8890
21	4331	4932	5561	6218	6902	7613	8351	9116
22	4458	5073	5714	6383	7078	7800	8547	9320
23	4576	5202	5855	6534	7238	7968	8723	9501
24	4685	5321	5983	6671	7383	8120	8879	9662
25	4785	5430	6100	6795	7513	8254	9017	9802
26	4877	5529	6205	6905	7627	8372	9137	9921
27	4959	5617	6298	7002	7727	8473	9238	10022
28	5033	5695	6380	7086	7813	8559	9323	10103
29	5098	5763	6451	7158	7885	8629	9391	10167
30	5154	5822	6510	7218	7943	8685	9443	10214
31	5202	5871	6559	7266	7989	8727	9479	10244
32	5242	5911	6598	7302	8021	8755	9501	10259
33	5274	5942	6627	7327	8042	8770	9510	10259
34	5298	5964	6645	7342	8051	8773	9504	10245

ϕ^0 \ q^2	0·36	0·38	0·40	0·42	0·44	0·46	0·48	0·50
0	0	0	0	0	0	0	0	0
1	635	699	768	843	923	1009	1102	1203
2	1266	1395	1533	1681	1840	2011	2196	2396
3	1892	2084	2289	2509	2746	3000	3275	3571
4	2510	2763	3034	3324	3635	3970	4329	4717
5	3116	3428	3762	4119	4502	4913	5354	5828
6	3708	4077	4471	4892	5343	5825	6341	6895
7	4284	4707	5159	5640	6153	6701	7287	7915
8	4842	5316	5821	6358	6929	7538	8187	8880
9	5380	5902	6456	7044	7668	8332	9038	9788
10	5896	6462	7061	7696	8368	9081	9836	10636
11	6389	6995	7636	8313	9028	9783	10581	11424
12	6858	7501	8178	8892	9645	10437	11272	12150
13	7302	7977	8688	9435	10219	11043	11908	12815
14	7719	8424	9164	9939	10751	11601	12490	13419

TABLE 4 (*cont.*)

$10^3 X(q^2, \phi)$

ϕ^0 \ q^2	0·02	0·04	0·06	0·08	0·10	0·12	0·14	0·16	0·18
0	3554	3877	4176	4472	4773	5085	5411	5756	6121
1	3554	3876	4175	4471	4772	5084	5410	5754	6120
2	3552	3874	4173	4468	4770	5081	5407	5751	6117
3	3549	3871	4169	4465	4765	5077	5402	5746	6111
4	3546	3867	4164	4459	4760	5070	5396	5739	6103
5	3540	3861	4158	4452	4752	5062	5387	5729	6092
6	3534	3854	4151	4444	4743	5052	5376	5718	6080
7	3527	3846	4142	4434	4732	5041	5363	5704	6065
8	3519	3836	4131	4423	4720	5027	5349	5688	6048
9	3509	3826	4120	4410	4706	5012	5332	5670	6029
10	3498	3814	4107	4396	4691	4995	5314	5650	6007
11	3487	3801	4092	4380	4674	4977	5294	5629	5984
12	3474	3787	4077	4363	4655	4957	5272	5605	5958
13	3460	3771	4060	4345	4635	4935	5248	5579	5930
14	3445	3754	4041	4325	4613	4911	5223	5552	5900
15	3429	3737	4022	4303	4590	4886	5196	5522	5868
16	3412	3718	4001	4280	4565	4859	5167	5491	5834
17	3394	3697	3978	4256	4539	4831	5136	5458	5798
18	3374	3676	3955	4231	4511	4801	5104	5423	5761
19	3354	3653	3930	4204	4482	4769	5070	5386	5721
20	3333	3630	3904	4176	4451	4736	5034	5348	5680
21	3311	3605	3877	4146	4420	4702	4997	5308	5637
22	3287	3579	3849	4116	4386	4666	4958	5266	5592
23	3263	3552	3820	4084	4352	4629	4918	5223	5546
24	3238	3524	3789	4050	4316	4590	4877	5178	5498
25	3212	3495	3757	4016	4279	4550	4834	5132	5448
26	3184	3465	3724	3980	4240	4509	4789	5084	5397
27	3156	3434	3691	3944	4201	4466	4744	5035	5345
28	3127	3402	3656	3906	4160	4422	4696	4985	5291
29	3097	3369	3620	3867	4118	4377	4648	4933	5235
30	3066	3335	3582	3827	4075	4331	4599	4880	5179
31	3034	3300	3544	3786	4031	4284	4548	4826	5121
32	3002	3264	3505	3744	3986	4235	4496	4771	5062
33	2968	3227	3465	3701	3939	4186	4443	4714	5002
34	2934	3189	3424	3656	3892	4135	4389	4657	4940
35	2898	3150	3383	3612	3844	4084	4334	4598	4878
36	2862	3111	3340	3566	3795	4031	4278	4538	4814
37	2825	3070	3296	3519	3745	3978	4221	4478	4750
38	2788	3029	3252	3471	3694	3923	4164	4416	4684
39	2749	2987	3207	3423	3642	3868	4105	4354	4618
40	2710	2944	3160	3373	3589	3812	4045	4291	4551
41	2670	2901	3114	3323	3536	3755	3985	4227	4483
42	2630	2857	3066	3272	3482	3698	3924	4162	4414
43	2588	2812	3018	3221	3427	3639	3862	4096	4345
44	2546	2766	2969	3168	3371	3580	3799	4030	4275
45	2504	2720	2919	3116	3315	3521	3736	3963	4204
46	2460	2673	2869	3062	3258	3460	3672	3896	4133
47	2416	2626	2818	3008	3200	3400	3608	3828	4061
48	2372	2577	2766	2953	3142	3338	3543	3759	3989
49	2326	2528	2714	2898	3084	3276	3478	3690	3916
50	2281	2479	2662	2842	3025	3214	3412	3621	3843
51	2234	2429	2609	2785	2965	3151	3345	3551	3769
52	2188	2379	2555	2729	2905	3087	3279	3480	3695
53	2140	2328	2501	2671	2844	3024	3211	3410	3621
54	2092	2277	2446	2614	2784	2960	3144	3339	3546

TABLE 4 (*cont.*)

$10^3 X(q^2, \phi)$

ϕ^0 \ q^2	0·20	0·22	0·24	0·26	0·28	0·30	0·32	0·34
0	6511	6930	7380	7866	8393	8967	9594	10281
1	6510	6928	7378	7864	8392	8965	9592	10279
2	6506	6924	7374	7859	8386	8959	9586	10272
3	6500	6917	7366	7851	8377	8950	9575	10260
4	6491	6908	7356	7840	8365	8936	9560	10244
5	6480	6896	7343	7826	8349	8919	9541	10223
6	6466	6881	7326	7808	8330	8898	9518	10197
7	6450	6863	7307	7787	8307	8873	9490	10167
8	6432	6843	7285	7763	8281	8844	9459	10133
9	6411	6820	7261	7736	8252	8812	9424	10095
10	6388	6795	7233	7706	8219	8777	9386	10052
11	6362	6767	7203	7674	8184	8738	9343	10006
12	6334	6737	7170	7638	8145	8696	9297	9956
13	6304	6704	7135	7600	8103	8651	9248	9902
14	6272	6669	7097	7559	8059	8602	9195	9844
15	6237	6632	7057	7515	8011	8551	9139	9783
16	6200	6593	7014	7469	7961	8496	9080	9719
17	6162	6551	6969	7420	7908	8439	9018	9652
18	6121	6507	6922	7369	7853	8380	8953	9582
19	6079	6461	6872	7315	7795	8317	8886	9508
20	6034	6413	6820	7260	7735	8252	8816	9433
21	5988	6363	6767	7202	7673	8185	8743	9354
22	5940	6312	6711	7142	7608	8116	8668	9273
23	5890	6258	6653	7080	7542	8044	8591	9190
24	5838	6203	6594	7016	7473	7970	8512	9105
25	5785	6146	6533	6950	7403	7894	8430	9017
26	5730	6087	6470	6883	7330	7817	8347	8928
27	5674	6027	6405	6814	7256	7737	8262	8836
28	5616	5965	6339	6743	7181	7656	8175	8743
29	5557	5902	6272	6671	7103	7574	8086	8648
30	5497	5837	6203	6597	7024	7489	7996	8552
31	5435	5771	6132	6522	6944	7404	7905	8454
32	5372	5704	6060	6446	6863	7316	7812	8354
33	5308	5635	5988	6368	6780	7228	7717	8254
34	5242	5566	5913	6289	6696	7138	7622	8152

ϕ^0 \ q^2	0·36	0·38	0·40	0·42	0·44	0·46	0·48	0·50
0	11037	11872	12798	13828	14981	16275	17736	19394
1	11035	11869	12795	13825	14977	16271	17731	19388
2	11027	11861	12785	13814	14965	16257	17715	19370
3	11014	11846	12769	13796	14944	16234	17690	19340
4	10996	11826	12746	13771	14916	16202	17654	19300
5	10972	11800	12718	13739	14881	16162	17608	19248
6	10944	11769	12683	13701	14837	16114	17553	19185
7	10911	11733	12643	13656	14787	16057	17489	19113
8	10874	11691	12597	13604	14730	15993	17417	19031
9	10832	11645	12545	13547	14666	15922	17337	18941
10	10785	11593	12489	13484	14596	15844	17250	18843
11	10734	11537	12427	13416	14521	15760	17156	18738
12	10679	11477	12360	13343	14440	15670	17056	18626
13	10620	11412	12289	13264	14353	15574	16950	18508
14	10557	11344	12214	13182	14262	15473	16838	18384

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

TABLE 4 (*cont.*)

$-10^3 Y(q^2, \phi)$

ϕ^0 \ q^2	0·02	0·04	0·06	0·08	0·10	0·12	0·14	0·16	0·18
0	0	0	0	0	0	0	0	0	0
1	58	60	63	65	67	69	71	73	76
2	115	121	125	130	134	138	142	147	151
3	173	181	188	195	201	207	214	220	226
4	230	241	250	259	268	276	285	293	301
5	287	301	313	324	334	345	355	366	376
6	345	361	375	388	401	414	426	438	451
7	402	421	437	452	467	482	496	511	525
8	459	480	499	516	533	550	566	583	599
9	515	540	560	580	599	617	636	654	673
10	572	599	622	643	664	685	705	725	746
11	628	658	683	706	729	752	774	796	818
12	684	716	744	769	794	818	842	866	890
13	740	774	804	831	858	884	910	936	962
14	796	832	864	893	922	949	977	1004	1032
15	851	890	923	955	985	1014	1044	1073	1102
16	906	947	982	1016	1047	1078	1109	1140	1171
17	960	1004	1041	1076	1109	1142	1175	1207	1240
18	1014	1060	1099	1136	1171	1205	1239	1273	1307
19	1068	1116	1157	1195	1232	1267	1303	1338	1374
20	1122	1171	1214	1254	1292	1329	1366	1403	1440
21	1174	1226	1270	1312	1351	1390	1428	1466	1504
22	1227	1280	1326	1369	1410	1450	1489	1529	1568
23	1279	1334	1382	1426	1468	1509	1550	1590	1631
24	1330	1387	1436	1482	1525	1567	1609	1651	1693
25	1381	1440	1490	1537	1581	1625	1668	1711	1753
26	1432	1492	1543	1591	1637	1682	1726	1769	1813
27	1482	1543	1596	1645	1692	1737	1782	1827	1872
28	1531	1594	1648	1698	1746	1792	1838	1883	1929
29	1580	1644	1699	1750	1799	1846	1893	1939	1985
30	1628	1693	1749	1801	1851	1899	1946	1993	2040
31	1675	1742	1799	1852	1902	1951	1999	2046	2094
32	1722	1790	1848	1901	1952	2002	2050	2099	2147
33	1768	1837	1896	1950	2002	2052	2101	2150	2198
34	1814	1884	1943	1998	2050	2100	2150	2199	2248
35	1859	1929	1989	2045	2097	2148	2198	2248	2297
36	1903	1974	2035	2090	2144	2195	2245	2295	2345
37	1946	2018	2079	2136	2189	2241	2291	2342	2392
38	1989	2062	2123	2180	2233	2285	2336	2387	2437
39	2031	2104	2166	2223	2276	2329	2380	2430	2481
40	2072	2146	2208	2265	2319	2371	2422	2473	2524
41	2113	2186	2249	2306	2360	2412	2464	2514	2565
42	2153	2226	2289	2346	2400	2452	2504	2554	2605
43	2192	2265	2328	2385	2439	2492	2543	2593	2644
44	2230	2304	2366	2423	2477	2529	2580	2631	2681
45	2267	2341	2403	2460	2514	2566	2617	2668	2718
46	2304	2377	2439	2496	2550	2602	2652	2703	2753
47	2340	2412	2474	2531	2584	2636	2687	2737	2786
48	2374	2447	2509	2565	2618	2670	2720	2769	2819
49	2408	2481	2542	2598	2651	2702	2752	2801	2850
50	2442	2513	2574	2630	2682	2733	2782	2831	2880
51	2474	2545	2605	2660	2712	2763	2812	2860	2909
52	2505	2576	2636	2690	2742	2791	2840	2888	2936
53	2536	2606	2665	2719	2770	2819	2867	2915	2962
54	2565	2635	2693	2746	2797	2846	2893	2940	2987

TABLE 4 (*cont.*)

$-10^3 Y(q^2, \phi)$

$\phi^0 \backslash q^2$	0·20	0·22	0·24	0·26	0·28	0·30	0·32	0·34
0	0	0	0	0	0	0	0	0
1	78	80	82	84	87	89	92	94
2	155	160	164	169	173	178	183	188
3	233	239	246	253	260	267	274	282
4	310	319	328	337	346	356	366	376
5	387	398	409	420	432	444	456	469
6	464	477	490	504	518	532	546	562
7	540	555	571	586	602	619	636	654
8	616	633	651	669	687	706	725	745
9	692	711	730	750	771	792	813	836
10	767	788	809	831	854	877	901	925
11	841	864	888	912	936	961	987	1014
12	915	940	965	991	1018	1045	1073	1102
13	988	1015	1042	1070	1098	1127	1157	1188
14	1060	1089	1118	1147	1178	1209	1241	1274
15	1132	1162	1193	1224	1256	1289	1323	1358
16	1203	1235	1267	1300	1334	1368	1404	1441
17	1273	1306	1340	1375	1410	1446	1484	1522
18	1342	1377	1412	1448	1485	1523	1562	1602
19	1410	1446	1483	1521	1559	1599	1639	1681
20	1477	1515	1553	1592	1632	1673	1715	1758
21	1543	1582	1622	1662	1703	1746	1789	1834
22	1608	1648	1689	1731	1773	1817	1862	1908
23	1672	1713	1755	1798	1842	1887	1933	1980
24	1735	1777	1820	1864	1909	1955	2002	2051
25	1796	1840	1884	1929	1975	2022	2070	2120
26	1857	1902	1947	1993	2040	2088	2137	2188
27	1916	1962	2008	2055	2102	2151	2202	2253
28	1975	2021	2068	2115	2164	2214	2265	2317
29	2032	2079	2126	2175	2224	2274	2326	2380
30	2087	2135	2183	2232	2282	2334	2386	2440
31	2142	2190	2239	2289	2339	2391	2444	2499
32	2195	2244	2293	2344	2395	2447	2501	2556
33	2247	2296	2346	2397	2449	2502	2556	2612
34	2298	2347	2398	2449	2501	2554	2609	2666

$\phi^0 \backslash q^2$	0·36	0·38	0·40	0·42	0·44	0·46	0·48	0·50
0	0	0	0	0	0	0	0	0
1	97	100	102	105	108	112	115	118
2	194	199	205	210	216	223	229	236
3	290	298	307	315	324	334	344	354
4	386	397	408	420	432	444	457	471
5	482	496	510	524	539	554	570	587
6	577	593	610	627	645	664	683	703
7	672	690	710	730	751	772	795	818
8	766	787	809	832	855	880	905	932
9	858	882	907	932	959	986	1015	1044
10	951	977	1004	1032	1061	1091	1123	1156
11	1042	1070	1100	1130	1162	1195	1230	1266
12	1132	1162	1194	1227	1262	1298	1335	1374
13	1220	1253	1287	1323	1360	1398	1438	1481
14	1308	1343	1379	1417	1456	1498	1540	1586

TABLE 5. DEGREE MEASURE OF $\theta = \phi - 2\alpha \arctan \dfrac{q \sin \phi}{1 - q \cos \phi}$, FOR $\alpha = 0.724745$ $(\gamma = 1.4)$

ϕ^0 \ q^2	0·06	0·12	0·18	0·24	0·30	0·36	0·42	0·48
10	5·35	2·44	−0·45	−3·53	− 6·91	−10·71	−15·05	−20·06
20	11·00	5·56	0·33	−5·00	−10·59	−16·52	−22·83	−29·54
30	17·19	9·86	3·13	−3·41	− 9·89	−16·36	−22·83	−29·28
40	24·10	15·56	8·11	1·22	− 5·28	−11·48	−17·40	−23·05
50	31·80	22·68	15·10	8·38	2·27	− 3·35	− 8·55	−13·40
60	40·30	31·09	23·76	17·48	11·94	6·97	2·45	− 1·68
80	59·50	51·08	44·81	39·70	35·36	31·57	28·22	25·21
100	81·12	74·14	69·18	65·25	61·99	59·19	56·74	54·57
120	104·49	99·21	95·56	92·72	90·40	88·42	86·70	85·18
140	129·05	125·53	123·14	121·31	119·81	118·55	117·45	116·49
160	154·35	152·60	151·41	150·51	149·78	149·16	148·67	148·16